with periodic boundary conditions that are extracted from a cycle of the directed graphs G_1, G_2, and G_3, respectively. For the same initial configuration $\vec{\xi}_i$ ($i = 1, 2, 3$), we can see in Figure 7 that the dynamics evolving under f_{30}^{∞} exhibit the Bernoulli shift with $\sigma = 1$ and $\tau = 3$, which means shift the symbolic sequence to the left by one pixel after every three evolution steps. In Figure 8 we can see that the dynamics evolving under f_{41}^{∞} exhibit the Bernoulli shift with $\sigma = -1$ and $\tau = 3$, which means shift the symbolic sequence to the right by one pixel after every three evolution steps. In Figure 9 we can see that the dynamics evolving under f_{110}^{∞} exhibit the Bernoulli shift with $\sigma = -2$ and $\tau = 3$, which means shift the symbolic sequence to the right by two pixels after every three evolution steps.

In fact, many complex Bernoulli shifts exist that are components of attractors of rules 30, 41, and 110. In order to illustrate this, we consider the finite symbolic sequences with the length denoted by L. For different L, we pick out one symbolic sequence as an initial configuration, which is extracted from one of the periodic attractors under rules 30, 41, and 110, respectively. We can see that complex Bernoulli shifts exist for each initial configuration, as illustrated in Tables 1 through 3. This also to some extent perfectly characterizes the complex shift dynamics of rules 30, 41, and 110.

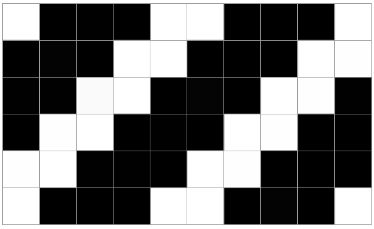

Figure 7. Bernoulli shift extracted every three steps from the dynamic evolution of f_{30}^{∞} with the initial configuration $\vec{\xi} = [0111001110]$.

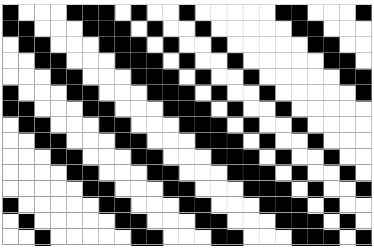

Figure 8. Bernoulli shift extracted every three steps from the dynamic evolution of f_{41}^{∞} with the initial configuration $\vec{\xi} = [1000111010010000011000 1]$.

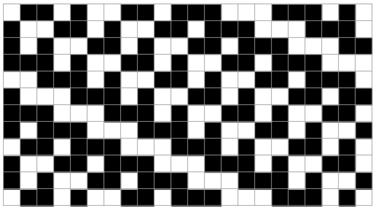

Figure 9. Bernoulli shift extracted every three steps from the dynamic evolution of f_{110}^{∞} with the initial configuration $\vec{\xi}$ = [0110100110111000111010].

L	Initial Configuration	Period	σ	τ
11	11100001110	154	1	56
12	000001111110	102	4	68
13	1000110000001	832	1	192
14	01111101110011	1428	1	1326
15	011000111000011	1455	−2	679

Table 1. List of complex Bernoulli shifts of rule 30 for different L ranging from $L = 11$ to $L = 15$.

L	Initial Configuration	Period	σ	τ
10	0000001001	40	2	24
11	00000000001	44	−1	12
12	000000100011	36	−1	15
13	0100000011000	117	1	18
14	00000000000001	28	2	12
15	000000000000001	60	−1	16
16	0010000100000000	176	−2	22

Table 2. List of complex Bernoulli shifts of rule 41 for different L ranging from $L = 10$ to $L = 16$.

L	Initial Configuration	Period	σ	τ
11	11101110011	110	1	50
12	011111000111	18	−2	3
13	1111111000100	351	1	189
14	01101111111110	91	−2	13
15	001111100110001	295	−3	118
16	1111100010011000	32	2	4
17	00111001100111110	578	−1	238
18	111000001101000011	81	2	9

Table 3. List of complex Bernoulli shifts of rule 110 for different L ranging from $L = 11$ to $L = 18$.

4. Concluding Remarks

This paper has demonstrated some complex shift dynamics of rules 30, 41, and 110 from the viewpoint of symbolic dynamics. By associating an interval map with these rules, it is shown that the interval map exhibits some degree of self-similarity. Based on directed graph theory, it is demonstrated that rules 30, 41, and 110 exhibit Bernoulli shifts and are topologically mixing on one of their own subsystems. Last but not least, for the finite symbolic sequences with periodic boundary conditions, many complex Bernoulli shifts are explored from the periodic attractors.

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