A Pit of Flowsnakes

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Vector replacement rules allow formulation of the flowsnake plane-filling curve as a bijection $\mathbb{Z} \longleftrightarrow \mathbb{Z}^2$ and as a surjection $\mathbb{Q} \longrightarrow \mathbb{Q}^2$. The $\mathbb{Q}$-function visits double and triple points predicted by Netto's theorem. Triple points belong to a relatively simple set, while double points resist easy classification. This paper formulates and analyzes the $\mathbb{Q}$-function. Methods applied to the flowsnake also apply to other plane-filling curves.

1. Introduction

The flowsnake belongs to a class of recursive functions—including the Hilbert curve and the dragon curve—that map from $\mathbb{R} \longrightarrow \mathbb{R}^2$. Plane-filling curves are some of the earliest examples of substitution systems, and interest in these "monsters" dates back to the time of Cantor, Peano, and Hilbert [1]. The map between spaces of different dimensions challenges the physical notion of a curve as "the locus of points obtained by moving a point moving continuously through space." Among the most interesting theorems about these curves, it was proven by Netto that no continuous map from $\mathbb{R} \longrightarrow \mathbb{R}^2$ could be injective [1]. To explore a consequence of this theorem, we will take time in Section 4 to prove theorems about the flowsnake double and triple points.

The flowsnake is a relatively new entry to the arena of plane-filling curves. It was reported early by Martin Gardner in a book about fractals, *Penrose Tiles to Trapdoor Ciphers* [2]. The original formulation is due to hacker prototype Bill Gosper. He has encouraged computation of plane-filling curves since 1972, when the topic occurred as item 115 in *Hakmem* [3]. Development of the $\mathbb{Q}$-function presented in Section 3 is independent from but influenced by recent collaboration between Bill Gosper, Julian Ziegler Hunts, and Neil Bickford [4].

The so-called $\mathbb{Q}$-function extends the exactly computable domain of the flowsnake from $\mathbb{Z}$ to $\mathbb{Q}$ by associating to every rational preimage a well-defined complex number. The map is a surjection $\mathbb{Q} \longrightarrow \mathbb{Q}^2$.

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Unlike the $\mathbb{Z}$-function, which determines a bijection $\mathbb{Z} \longleftrightarrow \mathbb{Z}^2$, the $\mathbb{Q}$-function visits some double and triple points in the complex plane. Exploring the rational values, we find that triple points occur only for certain preimages of the form $q = n \big/ 6d$, while double points occur over a wide variety of rational preimages.

Reformulation of the flowsnake changes the context of this curve from hobbyist plaything to research example. In so doing, we introduce techniques of formulation and analysis that apply to many other replacement functions.

2. Common Conception

A set of Lindenmayer substitution rules suffices to specify the flowsnake in one dimension. Often the Lindenmayer rules involve direction symbols such as + and −, but vector notation leads more readily to advanced formulations. In vector notation, the flowsnake requires a six-symbol lexicon that divides into two parts: $P[n]$ and $M[n]$, with $n = 0, 1, 2$. These symbols expand by a factor of seven upon every replacement. The replacement rules are

$$\begin{aligned} &P[n] \mapsto P[n], M\left[n-1\right], M[n], \\ &P\left[n+1\right], P[n], P[n]M\left[n+1\right]; \\ &M[n] \mapsto P\left[n+1\right], M[n], M[n], \\ &M\left[n+1\right], P[n], P\left[n-1\right], M[n]; \end{aligned} \tag{1}$$

where arithmetic on index n is evaluated modulo three.

The replacement rules of equation (1) determine a replacement function

$$\mathrm{INF}[x, j] = \mathrm{Part}[\phi[x], j], \tag{2}$$

where x is a symbol from the six-element lexicon, and $\phi[x]$ is the transformation of that symbol under the map of equation (1). The integer value j specifies a single part of each replacement image. A recursion of INF over the positive integers defines the function $\mathrm{FLSN}_{\mathbb{N}}$:

$$\begin{aligned} &\mathrm{FLSN}_{\mathbb{N}}\left[1\right] = \mathrm{INF}\left[P\left[0\right], 1\right] = P\left[0\right], \\ &\mathrm{FLSN}_{\mathbb{N}}[x] = \mathrm{INF}\left[\mathrm{FLSN}_{\mathbb{N}}\left[\frac{x+j}{7}\right], 7-j\right], \\ &\qquad x \in 7\mathbb{N} - j. \end{aligned} \tag{3}$$

As a fully numeric encoding, this sequence is A261180 in the OEIS [5]. In choosing fixed point $P\left[0\right]$, we overlook two other fixed points, $P\left[1\right]$ and $P\left[2\right]$. The sequences generated from these symbols occur as

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translates of the sequence defined by equation (3):

$$\begin{array}{ll} P[1] : \text{FLSN}_{\mathbb{N}}[3\times 7+x], & x\in\mathbb{N}, \\ P[2] : \text{FLSN}_{\mathbb{N}}[11\times 7+x], & x\in\mathbb{N}. \end{array} \tag{4}$$

According to the transformation properties of equation (1), we can use two of the fixed-point sequences to extend the domain to all integers:

$$\text{FLSN}_{\mathbb{Z}}[x]\begin{cases} \text{FLSN}_{\mathbb{N}}[x] & +x\in\mathbb{N} \\ \Pi\big[\,\text{FLSN}_{\mathbb{N}}[3\times 7-x]\,\big] & -x\in\mathbb{N} \end{cases},$$

where Π is a parity function that exchanges heads $P\longleftrightarrow M$. The $\mathbb{Z}$-function is also called a two-way infinite sequence.

The symbols $M[n]$ and $P[n]$ denote vectors, which change parity by complex conjugation. Replacement rules for changing the symbols into vectors in the complex plane are

$$\begin{array}{l} P[n]\mapsto\ +\ e^{i\pi 2n/3}, \\ M[n]\mapsto\ -\ e^{i\pi 2n/3}. \end{array}$$

These replacements transform a sequence of $\text{FLSN}_{\mathbb{N}}$ values into a sequence of complex numbers, which specifies a walk through the complex plane, as in Figure 1.

A finite depiction gives us an idea of what the flowsnake looks like. The pictures are intriguing and attractive, but we need to go beyond the commonly available graphics to truly understand the curve. Our foray into lesser-known territory takes us to the $\mathbb{Q}$-function.

Figure 1. The Gosper curve. These images are linear interpolations of the $\mathbb{Z}$-function with domain $\{0, 1, 2, \ldots 7^n\}$; $n = 0, 1, 2, 3$. Taking the complex conjugate of $P[n]$ and $M[n]$ changes the chirality of the curve.

3. The Flowsnake $\mathbb{Q}$-Function

This section extends the domain of FLSN from the integers to all rational numbers. We choose to write the $\mathbb{Q}$-function using radix-restricted power series expansions, which we also call septenary expansions. This approach is similar to the approach of Sagan, where n-ary expansions pair with iterated function systems [1]. We advance these calcu-

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lations by performing a periodic analysis that allows us to determine the convergence of a power series to a single complex value.

A potentially infinite sequence of integer digits locates any point on the continuous interval depicted in Figure 2. The $\mathbb{Z}$-function joins rotationally equivalent vectors throughout the plane, so we follow Gosper by introducing a conventional restriction to the domain [3–5]. Considering only the rational values $q \in [0, 1]$ introduces some deceptions along the boundary, but those deceptions can be silenced after the fact. We write the integer digits of a rational number in radix-7 as power series coefficients. The generating function is determined by a function-valued function

$$\text{QGF}\left[\frac{m}{o \times b^n}, x, b\right] = \Theta_1[m, n, x, b] + \frac{\Theta_2[m, o, x, b]}{1 - x^{\omega[o]}},$$

which satisfies

$$q = \text{QGF}\left[q, \frac{1}{b}, b\right].$$

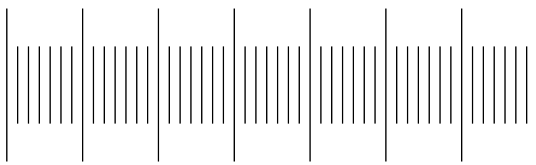

Figure 2. Hierarchical measurement system. Starting with seven divisions of an interval, each of seven intervals is further divided into seven parts, for a total of 49 intervals. Iterating this process recursively produces a 7^n geometric number line much like the 2^n number line on a United States ruler.

The Θ_1 and Θ_2 are polynomial functions of x that obey a restriction imposed by choice of radix, $b = 7$: in the power series expansion of $\text{QGF}[q, x, b]$, every expansion coefficient must belong to the set $0, 1, 2 \ldots 6$. Θ_1 captures only nonrepeating digits, while Θ_2 captures only repeating digits. Exact formulation of these functions is given in [6], which contains all subroutines that compose the well-defined $\mathbb{Q}$-function.

The difficulty arises with $\omega[o]$, the period of repetition. In a practical approach, we determine this integer by a recursive search that increments a temporary value for $\omega[o]$ until the actual value is found by comparing prime factorizations. Other sophisticated methods exist for determining $\omega[o]$, and the values of $\omega[o]$ can also be tabulated for a range of o.

After setting QGF, it becomes possible to expand the septenary power series

$$\text{QGF}[q, x, 7] = \sum_{n=1}^{\infty} a_n x^n.$$

As a shorthand, we can also write the expansion in septimal form:

$$q = 0\dot{;}_7 a_1 a_2 a_3 a_4 \ldots.$$

According to the $[0, 1]$ restriction, a_0 always equals zero.

Now recall that there exists a linear replacement function INF with a fixed point and inflation factor seven. This permits an interpretation of any septimal number as selection instructions that specify a sequence of replacement symbols from the six-element lexicon. Repeated application of INF, as in the function $\text{FLSN}_{\mathbb{N}}$, creates a hierarchical or branched topology between successive replacement images. Calling this tree a family tree, the sequence of replacement symbols determined by a septimal is also an unending lineage [7].

Terms of the lineage we denote p_n, and the fixed-point axiom is p_0. Under INF, a parent begets seven children. Of these seven children, the next parent in the lineage is

$$p_n = \text{INF}[p_{n-1}, a_n + 1],$$

where a_n is the n^{th} digit of the septimal expansion as above.

According to the restriction of the domain to rational numbers only, the sequence of a_n repeats with some period ω, possibly after ϕ nonrepeating initial values. The set of replacement symbols contains only six elements in total, so eventually the sequences a_n and p_n will exhaust all nonrepeating possibilities and begin to repeat in unison. There must exist some integers $j < k$ such that

$$p_{\phi+k\omega} = p_{\phi+j\omega} \,\&\&\, a_{\phi+k\omega} = a_{\phi+j\omega},$$

which implies a period of repetition $(k - j)\omega$. According to the mnemonic "when it repeats / then it completes," it should be possible to find a finite-term function that generates the infinite symbolic sequence as the coefficients of a power series expansion

$$g(x, q) = \sum_{n=0}^{\phi+j\omega-1} p_n x^{n+1} + \sum_{n=\phi+j\omega}^{\phi+k\omega-1} \frac{p_n x^{n+1}}{1 - x^{(k-j)\omega}}. \tag{5}$$

To this function we apply the map

$$p_n \mapsto \psi[\text{Total}[\{\text{INF}[p_n, m] : m \in 1, 2 \ldots a_{n+1}\}]],$$

where ψ denotes the transformation of equation (4). The complex-valued function $\tilde{g}(x, q)$ satisfies

$$\text{FLSN}_{\mathbb{Q}}[q] = \tilde{g}(\lambda, q),$$

where λ is the complex expansion factor of the substitution system, in our convention

$$\lambda = \frac{e^{-i\text{ArcCos}\left[5\sqrt{7}/14\right]}}{\sqrt{7}}.$$

Thoughtlessly introducing powers of such a complex number can lead to poor formatting for the functional output and slow computation. For this reason it is useful to define the real and imaginary parts of λ^n using two linear recurrences

$$\text{Re}[\lambda] = \frac{5}{14}, \text{Re}\left[\lambda^2\right] = \frac{11}{98};$$
$$\text{Im}[\lambda] = -\frac{\sqrt{3}}{14}, \text{Im}\left[\lambda^2\right] = -\frac{5\sqrt{3}}{98};$$
$$f[\lambda^n] = \frac{5}{7}f\left[\lambda^{n-1}\right] - \frac{1}{7}f\left[\lambda^{n-2}\right],$$

where f stands in for either function Re or Im. The $\mathbb{Q}$-function needs to repeatedly compute powers of λ. Algorithm timing improves when the values of $\text{Re}[\lambda^n]$ and $\text{Im}[\lambda^n]$ are saved as they are computed.

Improving the arithmetic leads to a number of positive outcomes. It becomes obvious that $\text{FLSN}_{\mathbb{Q}}[q] \in \mathbb{Q}^2$ for all rational q. The values of $\text{FLSN}_{\mathbb{Q}}[q]$ follow a simple reduction procedure to the form $a + bi$. The computational velocity, in complex values computed per second, compares favorably to the velocity of an algorithm that invokes magic simplification routines. It is worthwhile to optimize the arithmetic.

To describe the septenary expansion, we use a genealogical analogy. The space-filling property extends this analogy. $\text{FLSN}_{\mathbb{Q}}$ determines a diaspora where every lineage in the restricted domain converges to a point of rest within a two-dimensional space, the complex plane. In this metaphor, distant relatives sometimes meet at certain points within the diaspora. In Section 4, we study the so-called double and triple points.

4. Double and Triple Points

According to the theorem of Netto, any space-filling curve must visit some points in the complex plane multiple times. To observe this behavior in the flowsnake, we need only compute the images of certain rational values. We find these values by computer-aided exploration.

It is convenient to restrict exploration of function values to the images of rationals belonging to a well-defined set. For preliminary calculations, we choose domains of the form

$$S_d = \left\{ s = \frac{x}{d} : x \in 0, 1 \ldots d \right\}.$$

Evaluating $\text{FLSN}_{\mathbb{Q}}$ over all the values of S_d, we construct sequences that interpolate the flowsnake according to the natural ordering given by the < function. Except for S_7 and S_{49}, the images of S_d are unfamiliar, as Figure 3 shows.

Inspecting Figure 3, we make a number of observations. It is apparent that S_d and $S_{7\times d}$ bear a simple relation. The graphical image of $S_{7\times d}$ is obtained by scaling, rotating, and combining copies of the graphical image of S_d. The validity of this construction relates to the fact that $S_{7\times d} \supset S_7$ and $S_{7\times d} \supset S_d$. Practically, this construction leads us to redefine the domain sets

$$S_{r,n} = \left\{ s = \frac{x}{r 7^n} : x \in 0, 1 \ldots r \times 7^n \right\}.$$

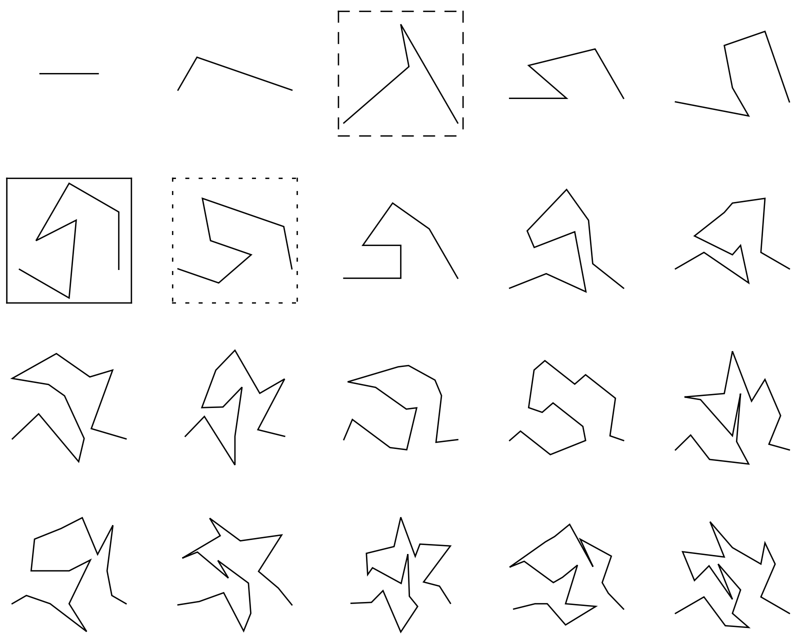

Figure 3. (*continues*).

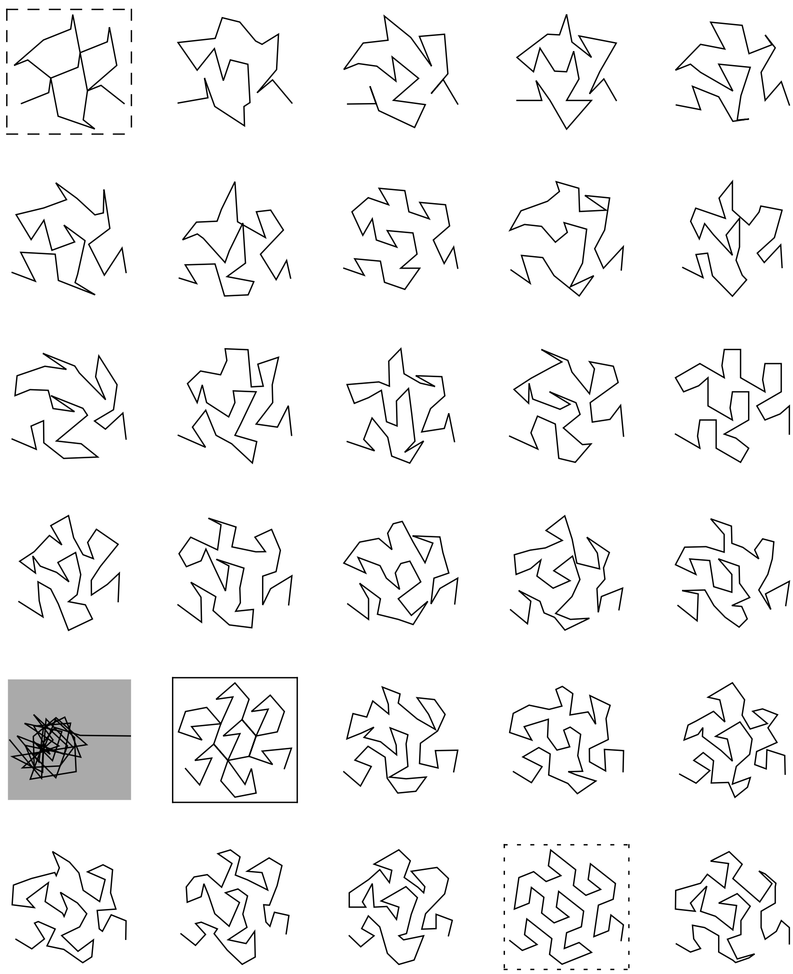

Figure 3. Graphical images of the flowsnake Q-function. Element $t_{i,j}$ of this graphics table corresponds to the set $S_{5(i-1)+j}$. The algorithm encounters a precision error at S_{41} and produces a scribble defect. The images most relevant to multipoint exploration correspond to S_6, S_{42}.

With the extra restriction that r does not contain seven as a prime factor, there is a bijection $S_d \longleftrightarrow S_{r,n}$.

Next we observe that the sets $S_{r,n}$ are mostly self-avoiding walks, with the exceptions being $S_{3,n}$, $S_{6,n}$, and $S_{27,n}$. The imminent exception is $S_{6,n}$, which is apparently the simplest domain sequence where triple points occur. Clearly this domain sequence deserves more investigation. Of the values in $S_{6,1}$, we are particularly interested in the subset

$$S_A = \left\{0, \frac{1}{6}, \frac{4}{6}, \frac{5}{6}, 1\right\},$$

with images

$$\text{FLSN}_{\mathbb{Q}}[S_A] = \left\{0, \frac{1}{6}\left(3 - i\sqrt{3}\right), \frac{1}{2}\left(1 + i\sqrt{3}\right), 1 + i\frac{\sqrt{3}}{3}, 1\right\}.$$

These are five of the six vertices of a hexagon in the complex plane centered around the point with preimage

$$q = 0;_7\overline{24} = \frac{14 + 4}{48} = \frac{18}{48} = \frac{3}{8},$$

and image

$$\text{FLSN}_{\mathbb{Q}}\left[\frac{3}{8}\right] = \frac{\left(1 + e^{\frac{i\pi}{3}}\right)\lambda - 2\lambda^2}{1 - \lambda^2} = \frac{1}{6}\left(3 + i\sqrt{3}\right).$$

By analysis of various lineages we arrive at the proposition that the missing point is obtained from the rational number

$$q = 0;_7 3\overline{5} = \frac{1}{7}\left(3 + \frac{5}{6}\right) = \frac{23}{42}.$$

For this $q \in S_{6,2}$, the function value is

$$\text{FLSN}_{\mathbb{Q}}\left[\frac{23}{42}\right] = \lambda e^{i\pi/3} - \frac{2\lambda^2}{1 - \lambda} = i\frac{\sqrt{3}}{3},$$

and indeed this is the missing hexagonal point. Now we form the complete vertex set

$$S_{V,0} = S_A \cup \left\{\frac{23}{42}\right\}.$$

A natural ordering for $S_{V,0}$ is provided again by the function <, so we can depict $\text{FLSN}_{\mathbb{Q}}[S_{V,0}]$ as a path that visits all vertices of a hexagon in the complex plane, as with the fourth tile in Figure 4.

To prove appearances and enumerate counting sequences, it is expedient to pass from $\text{FLSN}_{\mathbb{Q}}$ to a hybrid tiling system. The tiling substitution system is computationally fast, easy to analyze, and easy to construct. The construction involves marking one prototile with a path determined by $\text{FLSN}_{\mathbb{Q}}$ and specifying a function DEF for subdividing the two-dimensional space into nested configurations of tiles.

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Iteration of DEF will always produce another level of the hierarchy with a greater number of smaller tiles.

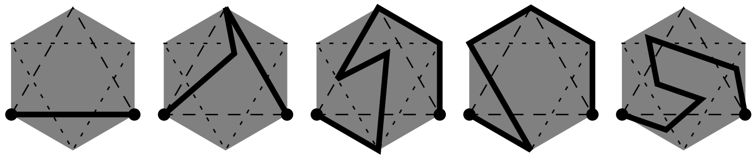

Figure 4. *Inscribed tiles*. From left to right, tiles corresponding to $S_{1,0}$, $S_{3,0}$, $S_{6,0}$, $S_{V,0}$, $S_{1,1}$. Endpoints at 0 and 1 are marked with a disk. Coincidence of various paths with vertices of the dotted and dashed triangles determines if the domain sets lead to double or triple points.

Create an axiomatic, hexagonal prototile by inscribing a path from $\mathrm{FLSN}_{\mathbb{Q}}$ onto a hexagonal tile bounded by the vertices $\mathrm{FLSN}_{\mathbb{Q}}\left[S_{V,0}\right]$. We call this tile $T\left[0\right]$. From this tile, we generate a tileset with an infinite number of tiles

$$\tau = \left\{\lambda^n T[\theta] : n \in 0, 1, 2 \ldots \infty;\ \theta \in 0, 1, 2\right\},$$

where $T\left[1\right]$ and $T\left[2\right]$ are obtained from $T\left[0\right]$ by a rotation of $2\pi / 3$ and $4\pi / 3$ radians. The function DEF subdivides every tile of the form $\lambda^n T[\theta]$ into seven tiles of the form $\lambda^{n+1} T[\theta]$. Formally, we write an expression similar to equation (2):

$$\mathrm{DEF}[x, i] = \mathrm{Part}[\phi_2[x], i],$$

where ϕ_2 is determined by a set of two-dimensional replacement rules. It is most expedient to write ϕ_2 in a pictorial language, as in Figure 5.

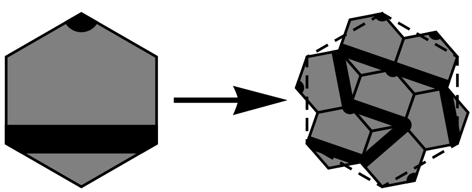

Figure 5. *Deflation replacement*. This pictorial replacement rule shows the preimage and image with path $\mathrm{FLSN}_{\mathbb{Q}}\left[S_{1,0}\right]$ inscribed on the tile. The other replacement rules are obtained by simultaneous rotation of image and preimage. The $\mathbb{Q}$-function allows inscription of paths from a domain set other than $S_{1,0}$.

If we merge the $\mathbb{Q}$-function over $\left[0, 1\right]$ and the $\mathbb{Z}$-function, we obtain a $\mathbb{Q}$-function defined for every rational because the $\mathbb{Z}$-function determines a placement of tiles at scale λ^0 throughout the plane. By this construction, we extend each set S in the restricted domain to a set $\tilde{S}$ in the unrestricted domain that obeys

$$S \subset \tilde{S} \subset \mathbb{Q}.$$

For the complete $\mathbb{Q}$-function, iteration of DEF yields a hierarchy of tilings related one to another by powers of λ. Each vertex of each tiling is visited by the path $\text{FLSN}_{\mathbb{Q}}[S_{V,0}]$ inscribed on an appropriately transformed tile. Clearly all vertices of all tilings are multipoints, but we make a stronger observation.

In a tiling by hexagons, the vertices are given by a union of two hexagonal point lattices, which are invariant under rotations of 0, $2\pi/3$, and $4\pi/3$ radians about the center of any hexagon. The endpoints of the curve on a tile at any level of the hierarchy belong to only one of these point lattices, regardless of location or orientation. When the domain of $\text{FLSN}_{\mathbb{Q}}$ is all rationals, the only points visited by the $S_{V,0}$ interpolation will be at least a double point or at least a triple point, depending on which of two point lattices they belong to, as in the interior of Figure 6.

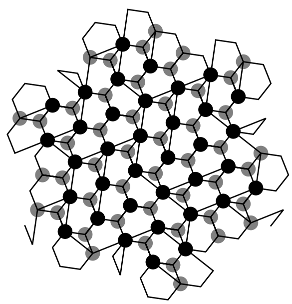

Figure 6. *Double and triple points*. In this image of $\text{FLSN}_{\mathbb{Q}}[S_{V,2}]$, double (triple) points are encircled by a gray (black) disk. Gosper's convention introduces deceptions along the boundary, where some triple points are inappropriately marked as double.

Now it is appropriate to review the assumptions of our venture. Is there any possibility that a union of some $S_{r,n}$ with $S_{V,0}$ will lead to additional self-intersections? Is it sufficient to look at only 49 output images or are 50 needed? As an answer to these questions, we prove the following theorems.

Theorem 1. In the limit of $n \longrightarrow \infty$, the set $\tilde{S}_{6,n}$ contains the preimages of all flowsnake $\mathbb{Q}$-function values that are also exact triple points.

Proof. This statement is equivalent to a statement about the tiling system determined above. Again we use the tileset marked with the path determined by $S_{V,0}$. The n^{th} iteration of DEF yields $\text{FLSN}_{\mathbb{Q}}[\tilde{S}_{V,n}]$, which is a subset of the values $\text{FLSN}_{\mathbb{Q}}[\tilde{S}_{6,n+1}]$. If repeated application of DEF generates a set of points containing all exact triple points, the theorem is certainly true.

The replacement rules of equation (1) require linear inflation factor seven, which in turn requires areal inflation factor seven. No finer division of the plane can exist. Iteration of DEF divides every hexagonal region into seven smaller hexagonal regions, creating a hierarchical structure. Levels of the hierarchy are scaled by a length factor λ such that

$$\lambda \times \lambda^* = \frac{1}{7},$$

where * denotes complex conjugation. This is exactly the correct scale factor so that the plane-filling deformation changes length into area. A bijection then relates a nested set of number line intervals of length $(1/7^n) l_0$ to a set of nested topological discs, also regions or tiles, with area $(1/7^n) A_0$.

At every n^{th} level of the hierarchy, the path carried by the tiles visits every point where three tiles join together. Equivalently, the set of values $\lim_{n\to\infty} \text{FLSN}_{\mathbb{Q}}[\tilde{S}_{V,n}]$ contains all points shared by three regions at some level of the hierarchy.

Let us assume that a value of $\text{FLSN}_{\mathbb{Q}}$ falls onto a hexagon vertex somewhere in the hierarchy. As discussed earlier, the point is at least a double/triple point. By convergence analysis, we show that these points are exactly double/triple points.

Place a ball of radius δ, $B[\delta]$, around the point in question. The intersection of $B[\delta]$ with values of $\text{FLSN}_{\mathbb{Q}}$ divides into three complementary subsets. In each subset, the preimages fall into three ranges of maximal length ϵ. If the point is at least triple, all three ranges are separated by a minimum length of σ. If the point belongs to the other hexagonal point lattice, then two of the subsets are nearly connected and separated from the third by a length σ. The images of all values in each subset approximate the point in question.

Now we proceed through the limit $\delta \longrightarrow 0$ by a discontinuous procedure. Define a sequence of radii

$$\delta_{n+1} = \frac{\sqrt{7}}{7} \delta_n,$$

which enacts the limiting procedure as $n \longrightarrow \infty$. To avoid complications introduced by fractal boundaries, choose δ_0 such that $B[\delta_0]$ falls well within the boundaries of three coincident hexagonal tiles at hierarchy level λ^m. According to the transformation properties of the hierarchy, $B[\delta_n]$ falls well within the boundaries of a trefoil at hierarchy level λ^{m+n}. Iterating n, we progress toward infinitesimal but discrete units of the tiling hierarchy. In the one-dimensional space, we have a sequence of ϵ_n and σ_n that obeys the recurrence equations

$$\epsilon_{n+1} = \frac{1}{7}\epsilon_n, \qquad 1 \geqslant \sigma_{n+1} \geqslant \sigma_n,$$

with limiting behavior

$$\lim_{n\to\infty} \epsilon_n = 0, \qquad \lim_{n\to\infty} \sigma_n = q_\sigma > 0,$$

where constant rational q_σ is simply the distance between the closest double- or triple-point preimages, measured along the number line. Then the point in question has exactly three or exactly two preimages, depending on the hexagonal point lattice that it belongs to. Figure 7 shows the first few iterations of a graphical convergence analysis for the triple point $5/7 + \left(4\sqrt{3}\right)\big/21i$.

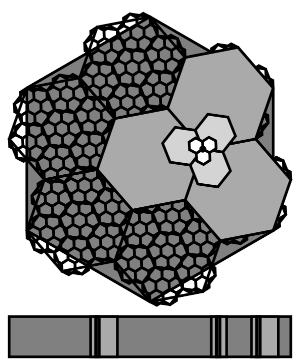

Figure 7. *Convergence analysis*. The two-dimensional graph shows a succession of hexagonal trefoils that converge toward $5/7 + \left(4\sqrt{3}\right)\big/21i$. These trefoils entirely enclose a ball $B[\delta]$ as the radius δ approaches 0. The one-dimensional graph shows a succession of intervals. The intervals have length less than ϵ, and they are separated by regions of length greater than σ. The intervals converge to three separate preimage values $13/42$, $31/42$, $37/42$.

Now it remains only to show that all triple points must fall onto a hexagon vertex somewhere in the hierarchy. According to the topology of the two-dimensional hierarchy, a point that does not fall exactly onto the intersection of three regions must fall exactly within the two-dimensional interior of a sequence of hexagonal regions or exactly onto the one-dimensional boundary between two hexagonal regions. There is no possibility for anomalous triple points.

By convergence analysis, we show that a point is exactly a triple point if and only if it is at least a triple point. Our set $\lim_{n\to\infty} \tilde{S}_{6,n}$ contains all rationals whose images are at least triple, therefore the theorem is proven. □

The left image of Figure 8 shows a flowsnake path that visits double points that also fall onto hexagon vertices somewhere in the hierarchy. The topology of the two-dimensional hierarchy allows double points to also exist along any of the one-dimensional boundaries separating two regions of the hierarchy. Along the one-dimensional boundaries, double points fall into a wide variety of domain sets.

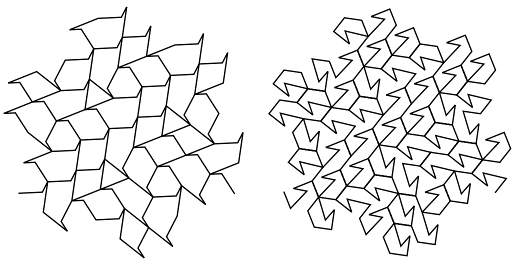

Figure 8. *Multipoint tilings*. The paths $\mathrm{FLSN}_{\mathbb{Q}}[S_{3,2}]$ and $\mathrm{FLSN}_{\mathbb{Q}}[S_{6,2}]$ visit points with more than one preimage. The interpolated path also divides the plane into a finite set of polygon tiles with exactly defined vertices. Both tilings must be aperiodic.

Theorem 2. Every set $S_{7^{\omega}-1,\delta}$—for all valid ω and δ—contains the preimage of a flowsnake $\mathbb{Q}$-function value that is also an exact double point.

Proof. In order to prove the theorem, we need only to show that it is possible to write a rational number q_β with a septimal expansion that meets two conditions:

1. The expansion digits must repeat according to an arbitrary period $\omega \in \mathbb{N}$ after δ nonrepeating initial values.
2. The expansion digits must determine a path that converges toward a boundary between two regions of the hierarchy.

Consider a number q_β with a radix-7 septimal such that all integer digits a_n are drawn from the set $\{4, 5\}$. Inspecting Figure 5, it is clear that the lineage for q_β obeys $p_n = p_0$ for all n because the fifth and sixth tiles, counting left-to-right, are both of type p_0.

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From Figure 5, we also see that $\mathrm{FLSN}_{\mathbb{Q}}\left[q_\beta\right]$ converges to the boundary. The fifth and sixth tiles in the deflation image are connected by a single edge of the preimage $E[0]$, which is an approximation of the fractal boundary. Under deflation, the edge obeys a Lindenmayer rule

$$E[n] \mapsto E[n]E[n+1]E[n], \tag{6}$$

where arithmetic on index n is evaluated modulo six, and the index n determines the direction of a vector in the complex plane.

The occurrences of $E[0]$ at positions one and three in the replacement image of $E[0]$ according to equation (6) are also boundary edges along the fifth and sixth tiles, respectively. Any number of replacements following the digits of q_β leads to a hexagon with an edge $E[0]$ along the boundary, so q_β converges to the boundary.

Now the only restriction on the digits of q_β is that each $a_n \in \{4, 5\}$ for all n. We are free to choose the sequence a_n to have δ nonrepeating initial digits followed by a sequence of digits that repeats with arbitrary period ω:

$$q_\beta \;=\; 0\dot{\,}_7 a_1 a_2 \ldots a_\delta \overline{a_{\delta+1} \ldots a_{\delta+\omega}}. \tag{7}$$

Every $q_\beta \in S_{7^\omega-1,\,\delta}$ that conforms to equation (7) converges to a regional boundary at hierarchy level λ^δ, thus the theorem is proven. □

According to Theorem 1, it is easy to locate triple points in the domain. Theorem 2 says the opposite about double points. The union of all valid sets $S_{7^\omega-1,\delta}$ is $\mathbb{Q}$, which implies that the $S_{r,n}$ may not be the best partitioning for capturing double points. Although we find some double points as in Figure 9, we leave the task of clearly formulating the double-point domain for future research.

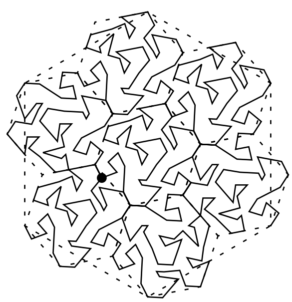

Figure 9. *A double point along the edge*. The path $\mathrm{FLSN}_{\mathbb{Q}}\left[S_{48,1}\right]$ visits a number of double points, including $\mathrm{FLSN}_{\mathbb{Q}}[11/112] = \mathrm{FLSN}_{\mathbb{Q}}[143/336] = \left(171 + 61\sqrt{3}\, i\right) \Big/ 546$, which is marked along the path by a disk. This particular double point falls between just two regions of hierarchy level λ^1.

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5. Surjection and Bijection

From Theorems 1 and 2, we already know that the $\mathbb{Q}$-function is not an injection. Now we prove another important property of the $\mathbb{Q}$-function.

Theorem 3. The flowsnake $\mathbb{Q}$-function is a surjection $\mathbb{Q} \longrightarrow \mathbb{Q}^2$.

Proof. We need to show that every rational image has at least one rational preimage, but we limit considerations to domain $[0, 1]$ without loss of generality.

Recall that DEF determines a hierarchical coordinate system in two dimensions, as depicted in Figure 10.

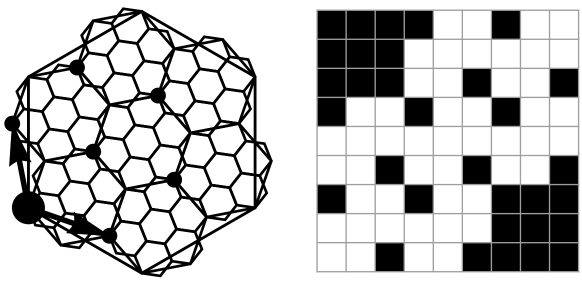

Figure 10. *Hierarchical* $\mathbb{Q}^2$ *coordinate system*. The hexagonal geometry in the left image transforms into the Cartesian geometry in the right image, where black cells remain undefined. The Cartesian geometry permits a ternary expansion.

A pair of rationals q_1, q_2 that falls into the hexagonal hierarchy also belongs to a subspace of a Cartesian hierarchy. It is easier to think in terms of the Cartesian coordinates. In this setting, vectors of the hexagonal hierarchy become the normal $\hat{x}$ and $\hat{y}$. We expand all co ordinates in radix-3:

$$q_1 = 0_3^{\cdot} a_1 a_2 a_3 \ldots, \qquad q_2 = 0_3^{\cdot} b_1 b_2 b_3 \ldots,$$

with the restrictions if $a_i = 2$ then $b_i \neq 2$ and vice versa. Both q_1, q_2 are rational, so the integer digits eventually repeat with period ω_1 and ω_2, which requires a combined period of $\omega = \text{LCM}[\omega_1, \omega_2]$. As in previous sections, the sequence of $\{a_i, b_i\}$ determines a sequence of parents p_i. The sequence p_i draws values from a finite lexicon. Eventually the possibilities become exhausted, and the sequences repeat in unison with period $k\omega$.

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Now, "when it repeats/then it completes," and, as in equation (5), we obtain at least one finite-term function for the point determined by $\{q_1, q_2\}$. Actually we can obtain a set of finite-term functions when $\{q_1, q_2\}$ is a multipoint, because each unique ternary expansion determines a unique finite-term function. There are many relations of the form

$$0;_3 a_1 a_2 02 = 0;_3 a_1 a_2 01\overline{2},$$

which make possible the existence of double and triple points.

In any case, each unique finite-term function obtained via two-dimensional expansion corresponds to exactly one finite-term function determined by equation (5), and thus to a rational number q. Finally, every element of $\mathbb{Q}^2$ has a preimage in $\mathbb{Q}$, so the $\mathbb{Q}$-function is a surjection. □

We would like to specify a set $\tilde{S}_{\text{DT}}$ containing all double and triple points, such that

$$\tilde{S}_{\text{inj}} = \mathbb{Q} \backslash \tilde{S}_{\text{DT}}$$

is a restricted domain over which the $\mathbb{Q}$-function is an injection and thus a bijection. According to the difficulties introduced by Theorem 2, we cannot yet complete the specification of $\tilde{S}_{\text{DT}}$ or $\tilde{S}_{\text{inj}}$. Without specifying these sets, we can still ask the exploratory question, what are the noteworthy domains where a flowsnake path avoids all double and triple points?

Let us propose one simple answer. Starting with set

$$S_{c,0} = \left\{\frac{3}{8}\right\},$$

we generate sets $S_{c,n}$ by repeated application of DEF, and sets $\tilde{S}_{c,n}$ by combination with the $\mathbb{Z}$-function. Convergence analysis shows that $S_{c,0}$ is a single point, the centroid of the region at hierarchy level λ^0. Convergence analysis is similar for all points in $\tilde{S}_{c,n}$, so we know that

$$\tilde{S}_{\text{inj}} \supset \tilde{S}_{c,n},$$

though we do not have an explicit formulation of $\tilde{S}_{\text{inj}}$. Figure 11 depicts $S_{c,2}$.

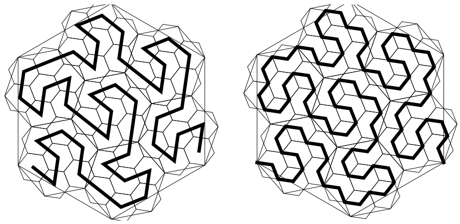

Figure 11. *Other flowsnake paths.* The left image depicts $\mathrm{FLSN}_{\mathbb{Q}}[S_{C,2}]$, which is a bijective path that visits only single points. The right image depicts $\mathrm{FLSN}_{\mathbb{Q}}[S_{C,2} \cup S_{1,2}]$, which is not a bijective path, because it alternates between single and double points.

Now we make an artistic observation that the path $\mathrm{FLSN}_{\mathbb{Q}}[S_{c,n}]$, like many other paths, is a branching double spiral. The meaning of this observation becomes more clear if we prune the path. The pruning procedure creates a new domain as follows. At each hierarchy level λ^{n+1}, take the six vertices from $\mathrm{FLSN}_{\mathbb{Q}}[S_{c,n+1}]$ that belong to the region of level λ^n containing $\mathrm{FLSN}_{\mathbb{Q}}[3/8]$ but do not belong to the region of level λ^{n+1} containing $\mathrm{FLSN}_{\mathbb{Q}}[3/8]$. Create a path by ordering the preimages according to the $<$ function. The output of this procedure is depicted in Figure 12.

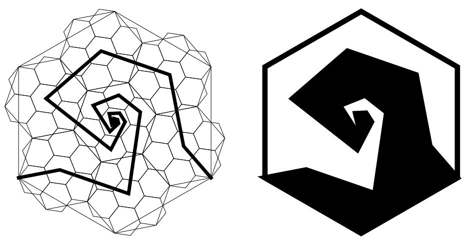

Figure 12. Flowsnake double spiral. The density of points distributed in space becomes greater as the spiral approaches the centroid of the λ^0 region at $\mathrm{FLSN}_{\mathbb{Q}}[3/8]$.

After pruning, we see clearly that the spiral first traverses downward through the hierarchy, ultimately converging on point $\mathrm{FLSN}_{\mathbb{Q}}[3/8]$. After reaching this point, the spiral begins to move upward through the hierarchy until it returns to level λ^0 at $\mathrm{FLSN}_{\mathbb{Q}}[1] = 1$.

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6. Conclusion

In this paper, we review and extend common formulations of the flowsnake plane-filling curve. We provide a $\mathbb{Q}$-function with well-defined values, which is equivalent to but fundamentally different from the formulation Gosper et al. use to determine triple points of the dragon curve [5, A260482]. Our formulation for the flowsnake finds double and triple points.

Though this article does present positive results regarding the multipoints of the $\mathbb{Q}$-function, the whole story remains untold. It remains a challenge to formulate a domain of the double points. Also, it remains to formulate—if it exists—a halting algorithm that computes the images of irrational preimages taken from a subset of the reals, such as $\mathbb{Q}\left[\sqrt{x}\right]$. Proofs in this paper do not rule out, for example, the possibility that multipoints exist in a hypothetical $\mathbb{Q}\left[\sqrt{x}\right]$-function.

In general, the analysis provided here extends to other space-filling curves whenever a few basic conditions are met. We assume the $\mathbb{Z}$-function can be written using a lexicon containing symbols that admit interpretation as vectors. The dragon curve also falls within this purview. By methods similar to those applied herein, we have also proven that the dragon curve $\mathbb{Q}$-function contains only single, double, and triple points in its range. Relaxing conditions, it is possible to generalize the analysis until it applies to most substitution systems.

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