Formula Processing on Physical Systems by Symmetry*

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Symmetry plays an important role in human reasoning about physical systems. Symmetry has been used as a constraint when deriving models for specific systems (as in dimensional analysis) as well as for physical laws. After reviewing human reasoning by symmetry in physics, this paper formalizes a symmetry-based reasoning. In dimensional analysis, only the scale symmetry for unit of measure of independent dimension is used to derive models. Symmetry-based reasoning is considered to be a generalization of dimensional analysis in the sense that it can take any symmetry and that it can deal with equations of physical laws as well as models of specific systems. Symmetry-based reasoning can be viewed as a symbolic version of finding fixed points where constraints are also given by symmetries represented by a symbolic level. In contrast to finding numerical fixed points of a given function, there are no systematic procedures for finding an invariant form. However, heuristics based on the strength of symmetry is given for efficiency. This symmetry-based reasoning system has been implemented as a symbol-processing system with a production system and a formula-processing system.

1. Introduction

Symmetry can be found almost everywhere; shapes of living things, artificial things, and regularity in physical phenomena, to mention only a few. A familiar example is reflection (vertical and horizontal) symmetry that is most appealling to perception. From Weyl [14], symmetry is:

> ". . . any point-set, has the peculiar kind of symmetry defined by the subgroup γ if it goes over into itself by the transformation of γ."

We take a broader view of symmetry:

> If we can define a mapping on an object and the image of the object is indistinguishable from the original object, then the object is said to have a symmetry defined by the mapping.

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We use the term "transformation" when the "mapping" is defined for specific objects of equations. The term mapping is used in a general context without specifying objects.

Objects that seem to be asymmetrical at first glance could be symmetrical when a mapping is properly defined. Finding symmetries, except those obvious to perception, require an elaboration. For example, symmetry in space-time (special relativity), required the very ingenious works of Lorentz [1] and Einstein [2].

One aspect of symmetry that has not yet been fully addressed is its function in human reasoning. Despite the fact that the environment is full of symmetry and that humans depend on symmetry for perception, memory, and reasoning, relatively little work on symmetry has been done in artificial intelligence. Work on the role of symmetry in reasoning about shape [3] and work on spatial reasoning [5] are exceptions to this general trend.

In this paper, we focus on a specific reasoning based on symmetry found in physics. We formalize the reasoning that we can trace in Einstein's 1905 paper [2] which emphasizes that the physical law should be expressed in a way invariant under its reference frame. Based on this principle, we specify and derive equations for physical laws by using symmetry as a guiding constraint.

Further, this investigation is not restricted to physics. If we take a broad definition of symmetry, such common sense reasoning as: "If it takes one hour to go from A to B, then it would also take one hour from B to A," can be recognized as examples of symmetry-based reasoning. In general, symmetry-based reasoning seems to be a weak method to which we resort when powerful knowledge, such as the dynamics of a particular system, are not readily available, as occurs with the theory of elementary particles.

Since symmetry by itself is not sufficient to completely specify equations for physical laws (as discussed in section 4), we can only propose plausible candidates of the formulas. We use a heuristic approach to make up for the insufficiency. The formula modification and refinement process also requires heuristics. The involvement of heuristic components requires the reasoning system to have a production system.

In section 2 we first review human reasoning in physics, especially by physicists, then discuss the extraction of reasoning with symmetry from Einstein's work on relativity. Stability analysis of complex systems and dimensional analysis are discussed as methods of symmetry-based reasoning. Section 3 presents primitives of symmetry-based reasoning. In section 4, the symmetry-based reasoning is formalized as a symmetry-based specification and derivation, and the example of specifying the equations for Black's law is presented.

2. Overview of symmetry-based reasoning

2.1 Review of human reasoning by symmetry in physics

The literature of physics includes several principles [6–13], at least four of which allow us to use symmetry in reasoning.

- Symmetry as characterization of underlying structure.
- Symmetry as guiding constraints.
- Symmetry as conserved property.
- Symmetry measurement in terms of entropy.

We use the symmetry as guiding constraints principle in the context of specifying and deriving equations for physical laws. In the following, we comment on each principle.

Symmetry as characterization of underlying structure. This principle is well-stated by Weyl [14]:

> "Whenever you have to deal with a structure endowed entity Σ try to determine its group of automorphisms, ... After that you may start to investigate symmetric configurations of elements, . . . "

This principle has demonstrated its power with group theory. Although this principle by itself can be used for the classification of phenomena and elaboration of purely mathematical structures, it has also been used in combination with the symmetry as conserved property principle to predict object behavior from object structure, as in crystallography and molecular chemistry.

Symmetry as guiding constraints. In this principle, symmetry is identified as a super law of the laws of nature. Symmetry, if given, can be used as constraints to specify or derive laws of nature, from Wigner [9]:

> "It is now natural for us to try to derive the laws of nature and to test their validity by means of the laws of invariance, rather than to derive the laws of invariance from what we believe to be the laws of nature."

There is an opposite position that regards symmetry as a derivable property [13]. Since we admit that the real process goes back and forth between underlying symmetry and physical law expressed as equations, we cannot dismiss this position. However, in this paper, we focus on the symmetry as guiding constraints principle.

It is worthwhile to mention that dimensional analysis in physics falls under this category. It uses scale symmetry to specify the dependency structure of a given physical system. Our approach in this paper can be seen as a generalization of dimensional analysis (see Example 2); we use not only scale symmetry but also other symmetries that are available.

Symmetry as conserved property. Two remarks follow on this principle. First, symmetry conservation can apply not only to causal mapping (i.e., a causal operator defines the mapping from a state at time 0 to a state at time t) but to other mappings, such as the mapping between different levels in a hierarchy, a deductive path in logic, mapping from information in a blueprint to the product, and so on. If we regard a characterization of a physical law as a mapping from underlying symmetry constraints to the symbolic expression of physical phenomena, the symmetry as guiding constraints principle assumes this symmetry as conserved property principle implicitly.

Second, symmetry is sometimes taken for granted to always be conserved. However, it is necessary for the associating transformation to be commutable with the causal operator, as is always the case when relating the law of conservation with underlying symmetry in classical physics. This will be formally stated in section 3.1.

Symmetry measurement in terms of entropy. Entropy is defined as a logarithm of W; the number of the equivalence subspace of microstate space [10]. If equivalence subspace is identified in an object, then a transformation of automorphism can be defined. The transformation permutes these equivalent subspaces, hence the object exhibits the corresponding symmetry.

This principle states the dynamics of symmetry, which is obtained by applying the second law of thermodynamics. The degree of symmetry of an object will evolve to be higher under a thermodynamical context. This consequence is a stronger claim than the symmetry as conserved property principle.

The symmetry as conserved property principle, applied to a specific mapping (many-to-one mapping), claims that the degree of symmetry does not decrease but may increase (see section 3.1 for details) as seen in the following Neumann–Minnigerode–Curie principle:

> "The group of the structure of a crystal is contained in the group of each of its physical properties. . . "

which was formally stated by Minnigerode in 1884 [11]. The mapping from the structure of a crystal to its physical properties is such a many-to-one mapping. This view of equating symmetry and entropy will be discussed more in section 3.1 from an information viewpoint of symmetry.

As an illustrating example, consider the problem of specifying figures by rotational symmetry around a center of rotation on a plane. Let a symmetry defined by a rotation of $2\pi/n$ around the center point be denoted by C_n.

Given the geometrical object of a hexagon, the symmetry as characterization of underlying structure principle can be used to characterize the object in terms of the geometrical operation of rotations on a plane. The hexagon has C_6 rotational symmetry and hence C_3, C_2 as the subgroup. This means that not only the set of rotations {0-rotation, $\pi/3$-rotation, $2\pi/3$-rotation, π-rotation, $4\pi/3$-rotation, $5\pi/3$-rotation} but its subsets; {0-rotation, $2\pi/3$-rotation, $4\pi/3$-rotation} and {0-rotation, π-rotation} have a group structure.

As an example of using the symmetry as guiding constraints principle, consider the problem of specifying figures by the constraint of C_6 rotational symmetry. Obviously, this symmetry cannot completely specify the hexagon object, but it can eliminate the possibility of a pentagon or octagon.

The symmetry as conserved property principle can be used to predict behavior. If the molecular structure has a geometry satisfying C_6 rotational symmetry, as does that of the Benzene ring, then its oscillation mode would have at least C_6 rotational symmetry. If the mapping from object to behavior is many-to-one (stated in Theorem 2), then behavior would have higher symmetry than that in the structure of the object using the measurement of symmetry in terms of entropy principle.

2.2 Examples of symmetry-based reasoning

This work is motivated by Einstein's work on relativity. Einstein's reasoning can be formalized as a symmetry-based reasoning that fully uses symmetry as a constraint. The reasoning can be applied not only to motion but also to other physical reasonings, or even common sense reasoning which is applied to a domain other than physics.

When one reviews Einstein's work on special and general relativity from the viewpoint of reasoning paradigm, one would realize they have a reasoning style in common: Both the special and general theory of relativity have two components, that is, the principle of relativity and the entity carrying physical meaning. As discussed in section 4 in the more general context of symmetry-based reasoning, we identify the former as symmetries and the latter as objects. In order to derive the symbolic forms either for the mapping defined in symmetries or for the object, symmetries are used as constraints that the object must satisfy.

One important target of symmetry-based reasoning is the analysis of such complex systems as nonlinear dynamical systems and probabilistic systems. The following example shows that the symmetry-based reasoning can be used for stability analysis of nonlinear systems.

Example 1 (Stability analysis of a nonlinear system) We consider an example of population dynamics. Two populations; x_i (population of the immne cell type i) and v_i (population of the virus cell type i) compete with each other. v_i can kill all types of x_i, however x_i can kill only the

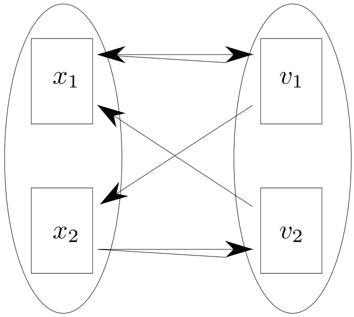

Figure 1. Two populations with asymmetric competitive interaction.

corresponding type v_i as shown in Figure 1. Population dynamics can be expressed by the following equations [15]:

$$\begin{aligned}
\dot{v}_1 &= v_1(b_1 - p_1 x_1) \\
\dot{v}_2 &= v_2(b_2 - p_2 x_2) \\
\dot{x}_1 &= k_1 v_1 - u_1(v_1 + v_2)x_1 \\
\dot{x}_2 &= k_2 v_2 - u_2(v_1 + v_2)x_2
\end{aligned}$$

where b_i, p_i, u_i, and $k_i (i = 1, 2)$ are parameters (positive constants). The stability condition for the virus population is already known [15]:

$$\frac{b_1 u_1}{k_1 p_1} + \frac{b_2 u_2}{k_2 p_2} < 1.$$

Suppose we want to derive the left-hand side of this inequality. In our context of symmetry-based reasoning, the following two components are given.

- O (Object to be specified): $f(b_1, p_1, k_1, u_1, b_2, p_2, k_2, u_2)$.
- T (Transformation of the symmetry): $b_i \to a_i b_i, p_i \to a_i p_i$ where a_i is any real number and $v_i \to g(v_j)$ denotes a substitution.

Thus, $O = T(O)$ gives the constraint:

$$(a_1 b_1, a_1 p_1, k_1, u_1, a_2 b_2, a_2 p_2, k_2, u_2) = f(b_1, p_1, k_1, u_1, b_2, p_2, k_2, u_2)$$

that is used to specify the symbolic form of $f(b_1, p_1, k_1, u_1, b_2, p_2, k_2, u_2)$. There are other transformations of the symmetry, such as exchange of the subscript 1 with 2, to further limit the candidate form for $f(b_1, p_1, k_1, u_1, b_2, p_2, k_2, u_2)$.

Symmetry is often used in geometry. One of the important differences between geometry and physics can be found in the scale symmetry. In physics the scale symmetry does not hold (as pointed out, for example,

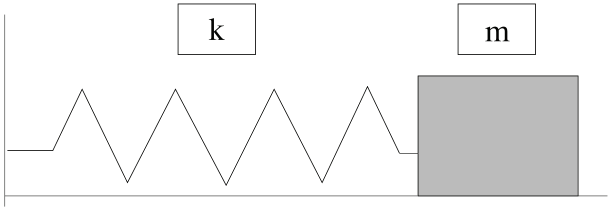

Figure 2. A simple mass-spring system without friction.

in [7]), that is, if you make a miniature system whose size is, say one-tenth that of the real system, then you cannot expect everything to be the same for the miniature as for the real one, even though all materials are the same except for size (known as scale effect). However, the scale symmetry for unit of measure of independent dimension must hold for physical systems as is known from Buckingham's Π theorem [16]:

> "If a physical system is described by $f(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are n variables that involve r basic dimensions, then the physical system will be described only by $n - r$ independent dimensionless products $\pi_1, \pi_2, \ldots \pi_{n-r}$. The equation describing the physical system is, thus, reduced to be $\phi(\pi_1, \pi_2, \ldots \pi_{n-r})$."

Again, it should be noted that Buckingham's Π theorem is understood as the result of applying the same reasoning as Einstein's theory of relativity: the representation of physical law (the equation describing a physical system in this case) must be represented as a form invariant under change of frame of reference (the system of dimension in this case). In fact, $\phi(\pi_1, \pi_2, \ldots \pi_{n-r})$ is such an invariant form since $\pi_1, \pi_2, \ldots \pi_{n-r}$ are invariant under scale change of unit of measure for each independent dimension.

Example 2 illustrates not only that the dimensional analysis can be done within the framework of symmetry-based reasoning but also that the symmetry of scale change of unit of measure for independent dimension can be treated as a constraint, similar to the other symmetries.

Example 2 (Dimensional analysis) Consider the motion of a simple mass-spring system without friction as shown in Figure 2.

The object to be specified is the formula of the system $f(t, k, m, x)$ where t is time (T), k is the spring constant (M/(T^2)), m is the mass of the pendulum (M), and x is the variance from the initial point (L) (dimension is indicated inside the parentheses). The given symmetries follow.

1. Scale symmetries due to the basic dimensions T, L, and M.
2. Phase translatory symmetry in terms of time t.

Symmetry 1 specifies the form $f(t, k, m, x)$ to: $f(t\sqrt{k/m})$. Symmetry 2 further gives the constraint: $f(t\sqrt{k/m}) = f((t + Tc)\sqrt{k/m})$.

Our method of specifying the equations of physical law can be viewed as a generalization of that in dimensional analysis.

3. Fromalization of symmetry-based reasoning for formula processing

3.1 Symmetry as a constraint

The following definition is merely a restatement of that by Weyl [14].

Definition 1 (Symmetry under a mapping) An object O has symmetry under a mapping T that can operate on O if $T(O) = O$.

With this definition of symmetry, a condition for symmetry as conserved property is stated as follows.

Theorem 1 (Condition for symmetry conservation) A symmetry of an object O defined by a mapping T is conserved under the mapping f if f is commutable with T.

Proof. $T(f(O)) = f(T(O)) = f(O)$. ■

Symmetry conserves under any mappings that commute with the mapping T. We have related symmetry to the mapping from an object to itself. Degree of symmetry may be defined by how many distinct mappings exist for the object; or equivalently, how many indistinguishable parts exist in the object. The more indistinguishable parts, the more distinct mappings that are defined by the permutation of distinguishable parts.

Theorem 2 (Condition for increase of degree of symmetry) The degree of symmetry of an object O defined by a mapping T may increase under mapping f if f is a many-to-one mapping.

Proof. Let o_i and o_j be distinct parts of O. If they are indistinguishable, that is, $o_i = o_j$, then $f(o_i) = f(o_j)$. However, even when they are distinguishable, that is, $o_i \neq o_j$, it may be the case that their images are indistinguishable $f(o_i) = f(o_j)$. In that case, the transformed object gains a degree of symmetry. ■

Definition 2 (Informational symmetry) If an object O must have a symmetry defined by a mapping T, then T can give information for an object O if $T(O) \neq O$.

Definition 3 (Order among symmetries) For two symmetries expressed by mappings T_i and T_j respectively, T_i is said to be stronger than T_j (denoted by $T_i > T_j$) if $T_j(T_i(O)) = T_i(O)$ for all the objects O but $T_i(T_j(O)) \neq T_j(O)$ for some objects O.

In other words, T_j cannot provide any information for all the objects O such that T_i cannot give information; that is, $T_i(O) = O$, but T_i can for some objects. T_i is said to be orthogonal to T_j if $T_j(T_i(O)) \neq T_j(O)$ for some objects O and $T_i(T_j(O)) \neq T_j(O)$ for some objects O.

A symmetry defined by a mapping T ceases to give information when the object exhibits symmetry of the same strength as T. In the language of group theory, $T_i > T_j$ means that the group corresponding to T_j is a subgroup of the group corresponding to T_i.

The specification process of equations for physical laws is the process of finding objects (formulas) whose symmetry is as close as possible to the strength of the symmetries given as constraints.

In summary, due to the absence of a symbolic version of the fixed point theorem, the specification and derivation process proceeds in the following manner, which is different from finding a numerical fixed point. For a given object O and a set of mappings $\{T_i\}$, pick up one mapping T and test if $O = T(O)$. If it is satisfied, then the symmetry corresponding to T is already embedded in O. If not, modify O so that $O = T(O)$ is satisfied. Continue the same procedure until all the mappings in $\{T_i\}$ satisfy $O = T(O)$. If the order of strength among $\{T_i\}$ is known, the mapping is tried by the order of strength.

Example 3. Since a regular triangle already has symmetry C_3, this symmetry does not provide any information to the regular triangle. However, both symmetries C_4 and C_6 will provide the information, for they require the figure to be a regular 12-gon and a hexagon, respectively. C_6 is a stronger symmetry than C_3 because C_3 cannot provide information for all the objects to which C_6 can provide information. C_4 is orthogonal to C_3 because C_3 can provide information for some objects to which C_4 cannot provide information. In this sense, the continuous symmetry C_∞, which specifies a circle, is the strongest possible symmetry.

In section 4.2. we look at Black's law of specific heat as an example of symmetry in a physical law.

3.2 Fixed point and symmetry

Our method of specifying equations for physical laws is a generalization of that in dimensional analysis. Buckingham's Π theorem can be generalized along the following lines (where the symmetry used is scale symmetry for unit of measure of an independent dimension and the invariant form is a dimensionless product in the Buckingham theorem).

If a formula is to be described with variables $x_1, x_2, \ldots, x_n$ and if the formula should have a symmetry that can be attained only by a form $F(x_1, x_2, \ldots, x_n)$, then the target formula can be reduced to the form $f(F(x_1, x_2, \ldots, x_n))$.

Example 4. If the given symmetry is the translatory symmetry such that the form should be invariant under the translation $x_1, x_2, \ldots, x_n \to x_1{+}c, x_2{+}c, \ldots, x_n{+}c$ where c is an arbitrary real number, then the unique form is $F(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a_1, a_2, \ldots, a_n$ are constants satisfying $a_1 + a_2 + \cdots + a_n = 0$. Therefore, the formula with this translatory symmetry can be reduced to the form $f(F(x_1, x_2, \ldots, x_n))$ by the preceding argument. (This form will be used in Example 5 to specify Black's law.) However, if the given symmetry is the permutation symmetry $x_i \leftrightarrow x_j$ ($v_i \leftrightarrow v_j$ denotes a permutation), then there are many forms satisfying the symmetry. For example, $F(x_i, x_j) = x_ix_j$, $F(x_i, x_j) = x_i + x_j$, and $F(x_i, x_j) = x_ix_j + x_i + x_j$.

The process of specifying equations for physical laws is the process of finding an object (formula) whose symmetry is as close as possible to the strength of the symmetries given as constraints. Due to the absence of a symbolic version of the fixed point theorem, the specification and derivation process is driven by observing how $T(O)$ differs from O and by using the equation $O = T(O)$. There is no continuous measure that indicates how close the current form is to the target solution in the search for the symbolic form in our procedure of symmetry-based reasoning. The discrete measure indicating how close the current form is to the target solution is the number of how many symmetries (out of the symmetries given as constraints) the current form satisfies. One heuristic to increase the efficiency of the search is that if the current solution satisfies the symmetry, then all the symmetries weaker than it can be disregarded.

4. Implementation of symmetry-based reasoning

4.1 Procedure of symmetry-based reasoning

Symmetry-based reasoning, in its most naive form, with respect to transformation T and object O may be formalized as follows.

Step 0 Given: Transformation T and some piece of information for an object O.

Step 1 Checking symmetry: If the object is given in symbolic form, then check whether the object satisfies the given symmetry. If $O = T(O)$, stop; otherwise go to Step 3.

Step 2 Proposing object: If object O does have symmetry, then propose a candidate formula of the object O by heuristics on the basis of the given list of symmetries T.

Step 3 Modifying object:

- Step 3.1 If $O = T(O)$ can derive a new transformation T' for the part of object O' then go back to Step 0 with these T' and O'.
- Step 3.2 If $O = T(O)$ include unknown parameters, solve the equation $O = T(O)$.
- Step 3.3 Modify O by heuristics so that $O = T(O)$ is satisfied. If there is not enough knowledge to do this, go back to Step 2 for a new proposal.

This symmetry-based reasoning has three different modes: deriving (and specifying) the symbolic form of the object, checking the symmetries of the object, and modifying the symbolic form of the object to satisfy the symmetries (generalization), depending on if the symbolic form of the target object is not known, already known, or partially known, respectively. These three modes are included in the reasoning procedure. The derivation (and specification) mode is carried out by using all of the steps. The checking mode is realized as Step 1 and is developed in both the derivation mode (after the concrete symbolic form is proposed at Step 2) and the modification mode. The modification mode occurs in Step 3, this mode is also used extensively by the derivation mode.

The rules of the symmetry-based reasoner can be divided into the following steps. The conditions in the left-hand side of the rules will be tested and fired in this order. The step numbers refer to the procedural steps described previously.

```
 Step 1.1 CheckSymmetry
 Step 1.2 Terminate
Step 2 ProposeFormula
  Step 3.1.1 DeriveConstraint
  Step 3.1.2 DeriveSymmetry
  Step 3.1.3 FocusIntObj
 Step 3.2 SolveEquation
 Step 3.3 ModifyObject
RefreshSymmetry
InitGoal
```

The rules of `CheckSymmetry` see if any of the given symmetries are satisfied by the given object using `SymQ` described in the Appendix. If all the given symmetries are satisfied by the object, then the rules of `Terminate` simply terminate the task. A simple symmetry check can be done using these two steps. We call this mode *symmetry identification*.

If none of the symmetries are satisfied by the object and there is knowledge available to propose modifications to it so that the unsatisfied symmetry may be satisfied afterwards, then the rules of `ProposeForm-`

`ula` will make those modifications. Among several symmetries that can propose modifications, permutation symmetries work in a unique manner. Permutation symmetries do not propose the modification, but rather derive constraints at Step 3.1.1 with `DeriveConstraint` and then internal symmetries found in that part of the current object are derived at Step 3.1.2 with `DeriveSymmetry`. Next, the permutation symmetries create a subgoal of specifying the part of the object using the new symmetries found at Step 3.1.3 with `FocusIntObj`. After these modifications, and after working memory elements have been preprocessed (done by `RefreshSymmetry`), the object is recast to the symmetry check process, returning control to Step 1.1 (`CheckSymmetry`) by the rule of `InitGoal`. We call this mode *symmetry-based specification.*

If the modification is just a specification of unknown parameters in the object, then the rules of `SolveEquation` in Step 3.2 are evoked after Step 1.2. In this step, the given symmetries are used for building the equation whose solution will specify the unknown parameter. Modification of the object is made by substituting the solution to the unknown parameter at Step 3.3 with `ModifyObject`. After this modification, the same process as that of symmetry-based specification follows. We call this mode *symmetry-based derivation.* In the following, we present sample sessions for these three modes.

4.2 Modes of symmetry-based reasoning

In this section, we only give examples of symmetry-based specification and symmetry-based derivation, since symmetry identification uses only part of the steps of symmetry-based specification. The examples of symmetry identification demonstrated in our reasoner include that the angular momentum is invariant under spatial rotation, that the quadratic form of $-(c^2t^2) + x^2 + y^2 + z^2$ is invariant under the Lorentz transformation, and that the Maxwell equation is invariant under the Lorentz transformation. The examples of symmetry derivation of parameters include that the specific heat from the symmetries used in Example 6 is the only unspecified parameter, the parameters in the Lorentz transformation, and the angle between spin and velocity for massless particle is zero (see [9]), which is more complex than those shown here.

Example 5 (Specification of the equation for Black's law) Black's law of specific heat can be stated as follows: If two entities, whose initial states are described by temperature t_1 and t_2, specific heat c_1 and c_2, and mass m_1 and m_2, are thermally coupled, then the final equilibrium temperature will be $t_f = (c_1m_1t_1 + c_2m_2t_2)/(c_1m_1 + c_2m_2)$.

In this example, the equation for Black's law, $t_f = (c_1m_1t_1 + c_2m_2t_2)/(c_1m_1 + c_2m_2)$, will be specified starting from scale symmetry (dimensional analysis), translatory symmetry of temperature, permutation symmetry of interacting entities, and permutation symmetry of specific heat and mass. When permutation symmetry exists in the list of symmetries in the input, three more steps are evoked after Step 2: `DeriveConstraint`, `DeriveSymmetry`, and `FocusIntObj`. These additional steps first derive constraints from permutation symmetry and then find the internal symmetry that this part of the object exhibits.

Example 6 specifies the equations for Black's Law when sufficient symmetries are given. Input and output from the symmetry-based reasoner is summarized in Appendix A.2.

Example 6 (Deriving specific heat in the equation of Black's law) Suppose we have the equation of Black's law, $t_f c_f(m_1 + m_2) = (c_1 m_1 t_1 + c_2 m_2 t_2)$, except for c_f. The remaining task is to derive c_f from a certain symmetry. A symmetry such as mass dilation: $m_i \to k m_i$ for k being any constant and entity permutation $(c_1, m_1, t_1) \leftrightarrow (c_2, m_2, t_2)$ cannot give any information to this object, since they are already embedded in it. However, temperature translation, $t_i \to t_i + c$ for c being any constant, will give information that specifies c_f to be $(c_1 m_1 + c_2 m_2)/(m_1 + m_2)$. This temperature translation can give the same information as that given by the conservation symmetry: $c_f(m_1 + m_2) = (c_1 m_1 + c_2 m_2)$. Input and output from the symmetry-based reasoner is summarized in Appendix A.2.

5. Related work

Viewed as a reasoning system, symmetry-based reasoning is distinct in that it is reasoning about *modeling* rather than reasoning about the *behavior* of given models. Although reasoning about behavior has been widely discussed and implemented, primarily in artificial intelligence, little has been done on the important topic of modeling.

Viewed as a law-discovery system [17], our implementation is in the same line as that of BLACK SYSTEM [18] in that it is theory-driven and uses a conservation concept. Although BACON 5 [19] uses symmetry in part to economize search, the implementation here relied on symmetry to form the base of the reasoning paradigm.

Although we address symmetry-based specification and derivation of equations for physical laws, we are fully aware not only that the final justification of the law should be against experimental data but that experimental data are required to bridge the gap between physical laws and symmetry.

Applying within a specific *symbol system* [20] would depend on the system's ability to identify symmetry in the representation adopted. Currently, it is possible for a computer symbol system to identify symmetry to some extent in symbolically expressed formulas; something which, up to now, has been possible only for a human symbol system using paper and pencil.

6. Conclusions

Among four types of symmetry, we formalized the symmetry as guiding constraints principle to automate reasoning in specifying and deriving equations for physical laws. It is demonstrated that if sufficient symmetry is given, it can specify, derive, and modify the equations of a physical law. Nevertheless, symmetry itself still needs to be found. There are many important symmetries which were not mentioned in this paper. In order to deal with complex systems, poweful symmetries such as renormalization group and Lee group, should be considered.

Symmetry-based reasoning implemented as a formula processing system on equations can be a poweful tool for analyzing complex systems. The symbolic equation serves as a powerful representation of knowledge that can be checked if it has a certain symmetry, solved if variables and constants are specified, transformed into canonical form, and so forth. In order for these operations to give physical implications, the syntax as well as semantics of the equation must be included in the knowledge of the system. In this paper, we implemented the syntax on the formula-rewriting system MATHEMATICA and the semantics on both the long-term and short-term memory of a production system.

Appendix

A. Formula-processing for symmetry-based reasoning

The formula-level operations required for the symmetry-based reasoning are: symmetry identification, equation building by symmetry, and equation solving. These operations are possible in commercial formula processing systems such as MATHEMATICA, MAPLE, MACSYMA, REDUCE, and so on. In our study, we used MATHEMATICA [21] for formula-processing. The intrinsic problem in symmetry identification comes from the lack of an ultimate canonical form of the equations, which makes it difficult for the system to identify two formulas as equivalent. Thus, even if the two forms are not identified as equivalent by the system, it may be simply because the system cannot identify the equivalence.

We used a production system [22] to implement the symmetry-based reasoning. The reasons we adopted the production system architecture are twofold. First, we formalized symmetry-based reasoning as based on a human reasoning paradigm rather than a computational algorithm. Second, symmetry-based reasoning requires much heuristic knowledge which is not explicitly included in the formula.

We implemented the production system with MATHEMATICA so that the formula-level operations mentioned previously could be done within the production system.

Like many other pattern-directed applications of MATHEMATICA, formula-processing for symmetry-based reasoning uses the chain of (conditional) rewriting of the formulas. Other than built-in functions such as `Solve` and `Eliminate`, the following components are implemented for symmetry-based reasoning: transformation of formula, symmetry identification, proposing formula by symmetry, and symmetry derivation from constraint. We present the syntax and examples for each of these components in the following sections.

A.1 Transformation of formula

The syntax of transforming an expression is:

`Trans_[exp_,{parameter-list}]`.

There are five types of transformations commonly used: `Transl`, `Perm`, `Dilat`, `Com`, and `ASCom`. The symmetry defined by the transformation will be expressed as `Trans_{parameter-list}`.

- `Transl[exp_,{x1,x2,...},c]` will translate the listed variables `x1,x2,...` to `x1+c, x2+c,...` in the expression specified by `exp_`.
- `Perm[exp_,{x1,x2,y1,y2,...}]` will permute `x1` with `x2`, `y1` with `y2`, and so on.
- `Dilat[exp_,{x1,x2,...},c]` will dilate the listed variables `x1,x2,...` into `c x1, c x2,...`.
- `Com[exp_,{x1,x2}]` will permute `x1` with `x2`.
- `ASCom[f_[x1,x2],{x1,x2}]` will make the function `f_[x1,x2]` into `1 - f_[1/x1,1/x2]`.

Other transformations which are specific to the problems will be described at each example. In the following examples, the input to the system is in boldface, the output from the system is not.

Example A.1.

```
Transl[-(c1 m1 t1) - (c2 m2 t2)
     + cf (m1 + m2) tf,{tf,t1,t2},c]

-(c1 m1 (c + t1)) - c2 m2 (c + t2)
     + cf (m1 + m2) (c + tf)

ASCom[f[m2/m1,c2/c1],{m2/m1,c2/c1}]

      m1  c1
1 - f[--, --]
      m2  c2
```

A.2 Symmetry identification

The symmetry of the formula can be mathematically identified by investigating the equivalence between the original formula and the formula after transformation as described previously. The syntax of the symmetry identification is:

```
SymQ[{},exp_,Trans_,,{parameter-list}]
```

which will identify if the expression `exp_` is equal to the expression after the transformation `Trans_[exp_,{parameter-list}]`. It will return `True` if the equality is identified, otherwise it returns `False`. As we have mentioned, it may return `False` even if the original expression and the transformed expression are equal after complicated reduction.

Example A.2.

```
SymQ[{},-(1 - (1 + (c2 m2)/(c1 m1))^(-1)
     + t2/((1 + (c2 m2)/(c1 m1)) t1))
     + tf/t1,Perm,{m1,c1,m2,c2}]
```

```
True
```

```
SymQ[{},f[c1, c2, m1, m2, tf, t1, t2],
     Transl,{tf, t1, t2}]
```

```
False
```

A.3 Proposing formula by symmetry

Symmetries can be used to propose possible forms that the target object may have. Since such proposals are made based on heuristics, the resulting object must be evaluated (see section 4.1 for the process) to see whether they satisfy all the given symmetries. The syntax of proposing possible forms is:

```
Trans_[prop,exp_,{parameter-list}].
```

Example A.3.

```
Transl[prop,f[c1, c2, m1, m2, tf, t1, t2],
     {tf, t1, t2}]
```

```
tf - (t1 (1 - f[c1, c2, m1, m2])
     + t2 f[c1, c2, m1, m2])
```

```
Dilat[prop,tf - (t1 (1 - f[c1, c2, m1, m2])
     + t2 f[c1, c2, m1, m2]),{c1, c2}]

                       c2
tf - (t1 (1 - f[1, --, m1, m2]) + t2
                       c1
      c2
f[1, --, m1, m2])
      c1
```

A.4 Symmetry derivation from constraint

Symmetries on the formula, such as permutation symmetry, can provide a constraint which can be used to derive internal symmetries (symmetry on part of the original formula), as seen in Example 5. The steps to carry out a symmetry derivation follow.

- First, eliminate irrelevant variables from the constraint (given by the symmetry).
- Then, use pattern-matching to search the symmetry for the extracted part of the original formula.
- After the internal symmetries are detected, they are recast to the symmetry-based reasoning system by creating a subgoal of specifying part of the original formula by the internal symmetries detected.

B. Examples of symmetry-based reasoning

B.1 Specification of equation by symmetry

Example B.1 (Specifying the equation of Black's law) Table 1 shows the symmetry-based specification of the equation for Black's law of specific heat. The step number in the leftmost column refers to the procedure in section 4.1.

B.2 Derivation of object parameters and properties by symmetry

Example B.2 (Deriving the specific heat in the equation of Black's law) Table 2 shows the symmetry-based derivation of the unknown parameter c_f.

Step	Item	Formula
	Variables	Symbol: Dimension
Step 0-i, given object O	final temperature initial temperature of entity 1 initial temperature of entity 2 mass of entity 1 mass of entity 2 specific heat of entity 1 specific heat of entity 2 unspecified function	t_f: θ t_1: θ t_2: θ m_1: M m_2: M c_1: $Q/(\theta M)$ c_2: $Q/(\theta M)$ $F(t_f, t_1, t_2, c_1, c_2, m_1, m_2) = 0$

Step0-ii, given symmetries	scale symmetry
	
	translatory symmetry with temperature
	permutation symmetry between entity 1 and 2
	permutation symmetry between mass and specific heat
		`Dilat{m1,m2}`,
		`Dilat{c1,c2}`,
		`Dilat{t1,t2,tf}`
		`Transl{tf, t1, t2}`
		
		`Perm{m1,m2,c1,c2,t1,t2}`
		
		`Perm{m1,c1,m2,c2}`

Step 2-i	Equation proposed by scale symmetry (dimensional analysis)	$f(t_f/t_1, t_2/t_1, c_2/c_1, m_2/m_1) = 0$
Step 2-ii	Equation proposed by translatory symmetry with temperature	$t_f - at_1 - (1-a)t_2 = 0$
Step 3.3	Modified by the equation of Step 2-i	$t_f/t_1 - g(c_2/c_1, m_2/m_1) - (1 - g(c_2/c_1, m_2/m_1))t_2/t_1 = 0$
Step 3.1.1-i	Equation by permutation symmetry $(t_1, c_1, m_1) \leftrightarrow (t_2, c_2, m_2)$	$t_f/t_1 - g(c_2/c_1, m_2/m_1) - (1 - g(c_2/c_1, m_2/m_1))t_2/t_1 = t_f/t_2 - g(c_1/c_2, m_1/m_2) - (1 - g(c_1/c_2, m_1/m_2))t_1/t_2$
Step 3.1.2-i, subgoal created for this symmetry	Constraint derived by the equation of Step 3.1.1-i	$1 - g(c_2/c_1, m_2/m_1) = g(c_1/c_2, m_1/m_2)$
Step 3.1.2-ii	Equation by permutation symmetry $c_i \leftrightarrow m_i$	$t_f/t_1 - g(c_2/c_1, m_2/m_1) - (1 - g(c_2/c_1, m_2/m_1))t_2/t_1 = t_f/t_1 - g(m_2/m_1, c_2/c_1) - (1 - g(m_2/m_1, c_2/c_1))t_2/t_1$
Step 3.1.2-ii, subgoal created for this symmetry	Constraint derived by Step 3.1.1-ii	$g(c_2/c_1, m_2/m_1) = g(m_2/m_1, c_2/c_1)$
Step 2-iii	Formula proposed by permutation symmetries of Steps 3.1.2-i and 3.1.2-ii	$g(x, y) = h(xy)$
Step 2-iv	Formula proposed by permutation symmetries of Step 3.1.2-ii and 2-iii	$h(x) = 1/(1+x)$
Step 1.2	All the symmetries are satisfied	$t_f - g(c_2/c_1, m_2/m_1)t_1 - g(c_1/c_2, m_1/m_2)t_2 = 0$ where $g(x, y) = 1/(1+xy)$

Table 1. Symmetry-based specification process for Black's law.

Step	Item	Formula
Step 0-i, given object O	given information for the object unspecified parameter	$t_f c_f (m_1 + m_2) - (c_1 m_1 t_1 + c_2 m_2 t_2) = 0$ c_f
Step 0-ii, given symmetries	Same as those of Step 0-ii in Table 1	
Step 3.1.1	Equation by temperature translation	$(t_f + c) c_f (m_1 + m_2) - (c_1 m_1 (t_1 + c) + c_2 m_2 (t_2 + c)) = t_f c_f (m_1 + m_2) - (c_1 m_1 t_1 + c_2 m_2 t_2)$
Step 3.2	Parameter derived by the equation of Step 3.1.1	$c_f = (c_1 m_1 + c_2 m_2)/(m_1 + m_2)$

Table 2. Symmetry-based derivation process of a parameter in Black's law.

References

[1] Lorentz, H. A., "Electromagnetic Phenomena in a System Moving with any Velocity Less than that of Light," in *The Principle of Relativity* (Dover Publications, New York, 1952).

[2] Einstein, A. "On the Electrodynamics of Moving Bodies," in *The Principle of Relativity*, (Dover Publications, New York, 1952).

[3] Leyton, M., "A Process-Grammar for Shape," *Artificial Intelligence*, **34** (1988) 213–247.

[4] Leyton, M., *Symmetry, Causality, Mind* (The MIT Press, Cambridge, MA, 1992).

[5] Liu, Y., and Popplestione R. J., "Symmetry Constraint Inference in Assembly Planning," *Proceeding of American Association of Artificial Intelligence*, **90** (1998) 1038–1044.

[6] Bridgman, P. W., *A Sophisticate's Primer of Relativity* (Wesleyan University Press, Middletown, 1962).

[7] Feynman, R. P., *The Character of Physical Law* (MIT Press, Cambridge, MA, 1965).

[8] Feynman, R. P. and Weingberg, S., *Elementary Particles and the Laws of Physics*, (Cambridge University Press, Cambridge, England, 1987).

[9] Wigner, P., *Symmetries and Reflections* (MIT Press, Cambridge, MA, 1967).

[10] Rosen, J., *A Symmetry Primer for Scientists* (John Wiley & Sons, New York, 1983).

[11] Brandmuller, J., "An Extension of the Neumann–Minnigerode–Curie Principle," *An International Journal of Computers & Mathematics with Applications*, **12B** (1986) 97–100.

[12] van Fraassen, B. C., *Laws and Symmetry* (Clarendon Press, Oxford, England, 1989).

[13] Froggatt, C. D. and Nielsen, H. B., *Origin of Symmetries*, (World Scientific, Singapore, 1991).

[14] Weyl, H., *Symmetry* (Princeton University Press, Princeton, NJ, 1952).

[15] Nowak, M. A., "Antigenic Diversity Thresholds and the Development of AIDS," *Science*, (1991) 963–969.

[16] Buckingham, E., "On Physically Similar Systems: Illustration of the Use of Dimensional Equations," *Physical Review IV*, **4** (1914) 345–376.

[17] Simon, H. A., *Models of Discovery* (Reidel, Dordrecht, 1977).

[18] Langley, P., *et al.*, *Scientific Discovery* (The MIT Press, Cambridge, MA, 1987).

[19] Langley, P., Bradshaw, G. L., and Simon, H. A., "BACON 5: The Discovery of Conservation Law," *Proceedings of IJCAI 81*, (1981) 121–126.

[20] Simon, H. A., *The Sciences of the Artificial* (The MIT Press, Cambridge, MA, 1969).

[21] Wolfram, S., *Mathematica: A System for Doing Mathematics by Computer* (Addison Wesley, Reading, MA, 1988).

[22] Newell, A. and Simon, H. A., *Human Problem Solving*, (Prentice-Hall, Englewood Cliffs, NJ, 1972).