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# Global Cellular Automata

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**Abstract.** Global cellular automata are introduced as a generalization of one-dimensional cellular automata allowing the next state of a cell to depend on a "regular" global context rather than just a fixed-size neighborhood. A number of well-known results for one-dimensional cellular automata are extended to global cellular automata.

## 1. Introduction

Cellular automata (CA) are models of complex systems in which an infinite lattice of cells is updated in parallel according to a simple local rule. A dynamical system on the lattice of cells is a continuous and shift-invariant function if and only if it can be specified by a CA.

We will generalize one-dimensional CA to provide for a "regular" global context, while still using simple transition rules specified by a simple finite transducer called an $\omega\omega$-sequential machine. Our global cellular automata (GCA) will retain most of the properties of CA and at the same time allow us to define many noncontinuous transition functions. An important special case is the possibility of using two or more "classical" CA rules in one dynamical system, with one of them selected to be applied for the whole or a part of the current configuration according to some "regular" conditions. Any "negative" result valid for one-dimensional CA—for example, any undecidability result—is, of course, also valid for GCA. We will not consider such problems. However, quite surprisingly, most "positive" results can be extended to GCA. Thanks to some techniques known for finite transducers, these extended results are proved rather easily. This is not so surprising when we note that in the simplified proof of the decidability of the injectivity problem for one-dimensional CA in [4], we actually have implicitly used GCA.

We assume that the reader is familiar with basic notions of automata and language theory; see, for example, [11]. In the section 2 we review the other necessary prerequisites and introduce $\omega\omega$-sequential machines ($\omega\omega$-SM). They are a special case (length-preserving) of $\omega\omega$-transducers from [9].

On the other hand, the sequential machines of [5, 7] are a special case of $\omega\omega$-SM, called simple $\omega\omega$-SM in section 6.

In section 3, we introduce GCA as a generalization of one-dimensional CA where the global CA function is defined by a single-valued and complete $\omega\omega$-SM. We show that the definition is effective, and we can test whether a given $\omega\omega$-SM has the required properties. We also show that GCA are indeed a generalization of CA; that is, every one-dimensional CA rule can be implemented by a complete and single-valued $\omega\omega$-SM. We give examples of GCA that cannot be implemented as CA and show some general techniques for constructing GCA—for example, by combining several one-dimensional CA working on disjoint domains of configurations.

In section 4 we study the well-known problems that are decidable for one-dimensional CA, in particular, injectivity and surjectivity. We extend these results to GCA.

In section 5 we study the limit sets and limit languages of GCA. Again, we succeed in extending the best-known results on one-dimensional CA to GCA.

Finally, in section 6 we consider the simple $\omega\omega$-SM, called sequential machines in [5, 7]. We briefly discuss the one-to-one correspondence between the simple $\omega\omega$-SM and sets of Wang tiles. We show the simple $\omega\omega$-SM from [5] that corresponds to the smallest known aperiodic set of 13 Wang tiles.

## 2. $\omega\omega$-finite automata and $\omega\omega$-sequential machines

First we recall the definition of the classical one-dimensional CA, with the neighborhood of a cell consisting of the cell itself and its $r$ neighbors to each side.

A CA is a triple $A = (S, r, f)$, where $S$ is a finite set of states, $r$ specifies the size of the neighborhood, and $f : S^{2r+1} \to S$ is the local function, also called the CA rule.

A *configuration* $c$ of the CA is a function $c : Z \to Z$ that assigns a state $S$ to each cell of the CA. The set of configurations is denoted $S^Z$. The local function $f$ is extended to the *global function*

$$G_A : S^Z \to S^Z.$$

By definition, for $c$, $d \in S^Z$, $G_A(c) = d$ if and only if $d(i) = f(c(i-r), c(i-r+1), \ldots, c(i+r))$ for all $i \in Z$.

The configuration space $S^Z$ is a product of infinitely many finite sets $S$. When $S$ is endowed with the discrete topology, the *product topology* on $S^Z$ is compact by Tychonoff's theorem [13, Theorem 5.13]. A subbasis of open sets for the product topology consists of all sets of the form

$$\{c \in S^Z \mid c(i) = a\}, \tag{1}$$

where $i \in Z$ and $a \in S$. A subset of $S^Z$ is open if and only if it is a union of finite intersections of sets of the form (1).

The shift $\sigma : S^Z \to S^Z$ is defined by $\sigma(c) = c'$, where $c'(i+1) = c(i)$ for each $i \in Z$. [14] gives the following characterization of CA as dynamical systems.

**Theorem 1.** *$G : S^Z \to S^Z$ is the global function of a CA if and only if it is shift-invariant and continuous.*

The set of bi-infinite words over $S$ is denoted by $S^{\omega\omega}$. For $c \in S^Z$ we denote by $c^{\omega\omega}$ the corresponding bi-infinite word in $S^{\omega\omega}$, similarly as for a set $C \subseteq S^Z$. If $d = \sigma^n(c)$, $n \geq 1$, then $d^{\omega\omega} = c^{\omega\omega}$. Hence any set $R \subseteq S^{\omega\omega}$ of bi-infinite words corresponds to a shift-invariant subset of $S^Z$.

Now we define our main tools, namely, the $\omega\omega$-finite automaton ($\omega\omega$-FA) [9] and the $\omega\omega$-sequential machine ($\omega\omega$-SM). The latter is a special case of the $\omega\omega$-transducer of [9] and a generalization of the sequential machine of [5]. Inputs of an $\omega\omega$-FA are bi-infinite words over an alphabet $S$ that can be viewed as shift-invariant classes of configurations in $S^Z$. A set $X \subseteq S^Z$ is said to be shift-invariant if $\sigma(X) = X$.

Finite automata that recognize sets of bi-infinite words were defined in [15] and studied in [2, 9, 10]. Here we use the definition from [9].

An *$\omega\omega$-FA* $A$ is a quintuple $(K, S, \delta, K_L, K_R)$, where

- $K$ is the finite set of states,
- $S$ is the input alphabet,
- $\delta : K \times S \cup \{\varepsilon\} \to 2^K$ is the transition function,
- $K_L \subseteq K$ is the set of left (accepting) states, and
- $K_R \subseteq K$ is the set of right (accepting) states.

An $\omega\omega$-FA $A$ can be represented by a diagram in the usual way, with the left states indicated by $L$ and the right states by $R$; see Figure 1.

A bi-infinite word $v$ is said to be recognized by $A$ if there is a mapping $Z \to K$, that is, a bi-infinite sequence of states

$$\ldots, q_{-2}, q_{-1}, q_0, q_1, q_2, \ldots$$

and a configuration $c$ in $v$ such that, for all $j \in Z$,

- $\delta(q_j, c_j) = q_{j+1}$, and
- there exist $m, n \in Z$, $m \leq j \leq n$, such that $q_m \in K_L$ and $q_n \in K_R$.

In other words, $v$ is said to be recognized by $A$ if there is a bi-infinite computation of $A$ on a configuration $c$ in $v$ such that there is a left state appearing arbitrarily early, a right state appearing arbitrarily late in the computation. Such a computation will be called an *accepting computation*.

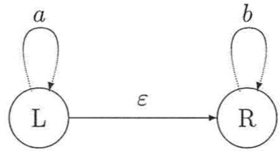



Figure 1: An $\omega\omega$-FA $A$.

The set of bi-infinite words recognized by $A$ is denoted $L(A)$. We call $L(A)$ an *$\omega\omega$-regular set*. Clearly, every $\omega\omega$-regular set over $S$ corresponds to a shift-invariant subset of $S^Z$.

Let $w \in \Sigma^\star$. By $w^\omega$ we denote the one-way infinite word ($\omega$-word) obtained by the infinite repetition of $w$. By ${}^\omega w$ we denote the reverse of $(w^R)^\omega$, that is, the infinite repetition of $w$ to the left. For example, the bi-infinite word ($\omega\omega$-word) of infinitely many $a$s followed by infinitely many $b$s is written as ${}^\omega ab^\omega$. For ${}^\omega aa^\omega$ we also write ${}^\omega a^\omega$ or $a^{\omega\omega}$.

**Example 1.** Let $A = (K, S, \delta, K_L, K_R)$ be an $\omega\omega$-FA, where $K = \{0, 1\}$, $S = \{a, b\}$, $K_L = \{0\}$, $K_R = \{1\}$, and $\delta$ is given in Figure 1. The set of bi-infinite words recognized by $A$ consists of the simple bi-infinite word ${}^\omega a b^\omega$.

The sets of finite (one-way), infinite, and bi-infinite words over $S$ are denoted by $S^\star$, $S^\omega$, and $S^{\omega\omega}$, respectively. Finite or one-way infinite words can be considered special cases of bi-infinite words in the following sense: A special quiescent symbol, usually 0, is specified such that a one-way infinite word ($\omega$-word) is a bi-infinite word with infinitely many consecutive quiescent symbols on the left end, and a finite word is a bi-infinite word with a finite consecutive nonquiescent subword.

In an $\omega\omega$-FA, a left (right) state that is not in a cycle can be changed into a nonleft (nonright) state without affecting the set of bi-infinite words recognized by the $\omega\omega$-FA. A state that cannot be reached from any left state or from which no right state can be reached is useless; it does not contribute to the recognition of any bi-infinite word. We say that an $\omega\omega$-FA is *reduced* if it satisfies the following conditions.

- Every left state is in a cycle.
- Every right state is in a cycle.
- Every state can be reached from some left state.
- From every state some right state can be reached.

Obviously, for any given $\omega\omega$-FA we can construct a reduced one that recognizes the same set of bi-infinite words.

An *$\omega\omega$-sequential machine* ($\omega\omega$-SM) is a 5-tuple $M = (K, S, \gamma, K_L, K_R)$, where

- $K$ is the set of states,
- $S$ is the input-output alphabet,

- $\gamma \subseteq K \times S \times S \times K$ is the transition relation,
- $K_L \subseteq K$ is the set of left states, and
- $K_R \subseteq K$ is the set of right states.

An $\omega\omega$-SM $M$ can be represented by a labeled directed graph with nodes $K$; an edge from node $q$ to node $p$ labeled $a, b$ for each transition $(q, a, b, p)$ in $\gamma$; the nodes in $K_L$, indicated by $L$; and the nodes in $K_R$ indicated by $R$.

Machine $M$ computes a relation $\rho(M)$, called an $\omega\omega$-SM relation, between bi-infinite sequences of configurations $S^Z$. Configurations $x$ and $y$ are in relation $\rho(M)$ if and only if there is a bi-infinite sequence $q$ of states of $M$ such that, for every $i \in Z$, there is a transition from $q_{i-1}$ to $q_i$ labeled by $x_i$,$y_i$ and there exist $m, n \in Z$ such that $m \leq i \leq n$, $q_m \in K_L$ and $q_n \in K_R$.

We give the closure and decidability results for an $\omega\omega$-regular set, which will be useful later. The following theorems immediately follow from [9, Corollary 2.6] and its proof.

**Theorem 2.** *The family of $\omega\omega$-regular sets is effectively closed under boolean operations.*

By modifying the proof for the closure of $\omega\omega$-regular sets under intersection, we get the following.

**Theorem 3.** *If $\rho$ is an $\omega\omega$-SM relation and $R$ is an $\omega\omega$-regular set, then the restriction $\rho_M = \{(u, v) \mid u \in R, (u, v) \in \rho\}$ of $\rho$ to $R$ is effectively an $\omega\omega$-SM relation.*

Since our $\omega\omega$-SM is a special case of the $\omega\omega$-transducer of [9], we have the following special case of Theorem 2.2 of [9].

**Theorem 4.** *The family of $\omega\omega$-regular sets is effectively closed under $\omega\omega$-SM relations.*

**Theorem 5.** *Given $\omega\omega$-FA $A, B$ it is decidable whether*

**(a)** $L(A) = \emptyset$

**(b)** $L(A) = S^Z$

**(c)** $L(A) = L(B)$

**(d)** $L(A) \subseteq L(B)$

*Proof.* Assume $A$ is reduced. Clearly, $L(A) \neq \emptyset$ if and only if there is a path from a state in $K_L$ to a state in $K_R$, which is easy to test; (b), (c), and (d) follow from (a) and Theorem 2. ■

A relation $R \subseteq S^Z \times S^Z$ is called *shift-invariant* if $(c, d) \in R$ if and only if $(\sigma(c), \sigma(d)) \in R$; $R$ is called *strongly shift-invariant* if $(c, d) \in R$ if and only if $(\sigma^i(c), \sigma^j(d)) \in R$ for all $i, j \in Z$. Clearly, every $\omega\omega$-SM defines a shift-invariant relation on $S^Z$. Note, however, that a relation on bi-infinite words over $S$ corresponds to a strongly shift-invariant relation on $S^Z$. Hence, two $\omega\omega$-SM might be equivalent on bi-infinite words, that is, on $S^{\omega\omega}$, without being equivalent on $S^Z$.

The proof of Lemma 2.4 of [9] is constructive. Thus we have the following representation lemma for $\omega\omega$-regular sets.

**Lemma 1.** *A set of bi-infinite words is $\omega\omega$-regular if and only if it can be presented by $D_1^R F_1 \cup D_2^R F_2 \cup \cdots \cup D_n^R F_n$, where $D_1, ..., D_n, F_1, ..., F_n$ are $\omega$-regular sets and $D^R$ denotes the reversal of $D$. Given an $\omega\omega$-FA $A$, such a canonical expression for $L(A)$ can be constructed.*

## 3. Global cellular automata

Now we are ready to introduce our main definition.

**Definition 1.** A global cellular automaton (GCA) is an $\omega\omega$-SM $M = (K, S, \gamma, K_L, K_R)$ that is

**(a)** complete, that is, $\mathrm{dom}(\rho(M)) = S^Z$; and

**(b)** single-valued, that is, $\rho(M)$ is a function.

Note that for every $\omega\omega$-SM $M$, $\rho(M)$ is shift-invariant.

The (global) function defined by GCA $M$ is denoted $G_M$. That is, $G_M : S^Z \to S^Z$, $G_M(c) = d$ if and only if $(c, d) \in \rho(M)$. It follows from Theorem 1 that any GCA function that is not a global CA function cannot be continuous.

Now we show that the definition of GCA is effective; that is, given an $\omega\omega$-SM, we can test whether $M$ is a GCA.

**Lemma 2.** *Given an $\omega\omega$-SM $M = (K, S, \gamma, K_L, K_R)$, it is decidable whether $M$ is complete, that is, whether the domain of $M$ is $S^Z$.*

*Proof.* The domain of every $\omega\omega$-SM is clearly a shift-invariant subset of $S^Z$, corresponding to an $\omega\omega$-regular set $R$. By omitting the outputs of $M$, we can easily construct an $\omega\omega$-FA $A$ such that $L(A) = R$. By Theorem 5 we can test whether $L(A) = S^Z$. ∎

Using the terminology of $L$-system theory, we call *coding* on $S^\star$ a letter-to-letter morphism on $S^\star$; that is, $c : S^\star \to S^\star$ such that $c(a) \in S$ for each $a \in S$.

We recall that $c^{\omega\omega}$ denotes the bi-infinite word over alphabet $S$ corresponding to $c \in S^Z$ (and to all its shifts). In [9] it has been stated that the Nivat theorem for finite transducers [3] can be restated for $\omega\omega$-finite transducers. For $\omega\omega$-SM, we have the following simple Nivat-like representation.

**Theorem 6.** *Let $\pi \subseteq S^{\omega\omega} \times S^{\omega\omega}$. There exists an $\omega\omega$-SM $M$ such that $R(M) = \pi$ if and only if there effectively exists an $\omega\omega$-regular set $R$ and codings $g, h : S^{\omega\omega} \to S^{\omega\omega}$ such that $\pi = \{(g(w), h(w)) \mid w \in R\}$.*

As an auxiliary tool we will use a one-way $\omega$-SM. An $\omega$-SM $M$ has initial states rather than left states. The definition of the relation $\rho(M)$ on $S^\omega$ is an obvious modification of the definition of the relation defined by an $\omega\omega$-SM.

**Lemma 3.** *Given a (one-way) $\omega$-SM $M$, it is decidable whether $M$ is single-valued (on $S^\omega$).*

*Proof.* The family of $\omega$-SM relations is clearly closed under composition and inversion. Hence, we can construct an $\omega$-SM $N$ such that $\rho(N) = (\rho(M))^{-1} \circ \rho(M)$. Clearly, $\rho(N)$ is a restriction of identity if and only if $M$ is single-valued. By Lemma 1 there effectively exist an $\omega$-FA $A$ and codings $g, h$ such that $\rho(N) = \{(g(w), h(w)) \mid w \in L(A)\}$. Clearly, $\rho(N)$ is a restriction of identity if and only if $g(w) = h(w)$ for all $w \in L(A)$. The latter condition is easy to test. ∎

Let $M$ be an $\omega\omega$-SM. Clearly, the single-valuedness of $R(M)$ is a necessary condition for the single-valuedness of $\rho(M)$. However, it is not sufficient; consider, for example, $M$ that defines the union of the identity and shift $\sigma$ on $S^Z$. We will use the $\omega$-regular sets (sets of one-way infinite strings) to test the single-valuedness of $\omega\omega$-SM.

**Lemma 4.** *Given an $\omega\omega$-SM $M$, it is decidable whether $\rho(M)$ is single-valued.*

*Proof.* We will construct a (one-way) $\omega$-SM $\hat{M}$ so that $\hat{M}$ is single-valued on $(S \times S)^\omega$ if and only if $M$ is single-valued on $S^{\omega\omega}$. Let $M = (K, S, \gamma, K_L, K_R)$. We first construct $\omega$-SM $M' = (K \times K, S \times S, \gamma', I, F)$, where $I = \{(q, q) \mid q \in K\}$ is the initial set of states, $F = K_R \times K$ is the set of final (right) states, and $((p, q), (a, b), (a', b'), (p', q')) \in \gamma'$ if and only if $(p, a, a', p') \in \gamma$ and $(q', b, b', q) \in \gamma$. Clearly, $M'$ simulates a computation of $M$ on an $\omega$-word with "two tracks" obtained by folding a bi-infinite word over $S$. However, $M'$ tests only the condition for the right states ($K_R$). In order to test the condition for the left states ($K_L$), we restrict the relation $\rho(M')$ to the $\omega$-regular set $R$ defined by $\omega$-FA $A = (K \times K, S \times S, \gamma_A, I, F_A)$, where $I$ is as above, right states are defined by $F_A = K \times K_L$, and the transition relation $\gamma_A$ is as follows: $((p, q), (a, b), (p', q')) \in \gamma_A$ if and only if for some $a', b' \in S$ $(p, a, a', p') \in \gamma$ and $(q', b, b', q) \in \gamma$. Similarly, as for bi-infinite words (Theorem 3), $\omega$-SM relations are clearly closed under $\omega$-regular sets, so there effectively exists an $\omega$-SM $\hat{M}$ with the properties described above. By Lemma 1 we can test whether $\hat{M}$ is single-valued. ∎

**Corollary 1.** *Given an $\omega\omega$-SM $M$, it is decidable whether $M$ is a GCA.*

*Proof.* By Lemmas 2 and 4. ∎

**Example 2.** $\omega\omega$-SM $M_1$, shown in Figure 2, is not single-valued; starting from any position $k \in Z$ we can map either every odd 1 to 0 and every even 1 to 1 or vice versa, so $M_1$ is two-valued.

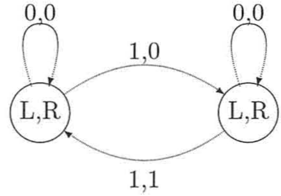



Figure 2: An $\omega\omega$-SM $M_1$.

**Lemma 5.** *Every one-dimensional CA (with arbitrary neighborhood $r$) can be implemented by a complete, single-valued (simple) $\omega\omega$-SM.*

*Proof.* CA $A = (s, r, f)$ can be simulated by $\omega\omega$-SM $M$ with $2r+1$ states. For simplicity we show the construction for $r = 1$. We construct $M = (S \times S \times S, S, \gamma, S, S)$, where $(p, q, r), q, f(p, q, r), (q, r, s)) \in \gamma$ for each $p, q, r, s \in S$. $M$ is nondeterministic but clearly single-valued, and complete. It is easy to verify that $G_M(c) = G_f(c)$ for all $c \in S^Z$.

Note that $M$ is a simple $\omega\omega$-SM according to the terminology introduced in section 2. ∎

**Theorem 7.** *The family of the global functions of one-dimensional CA is properly included in the family of the functions defined by GCA.*

*Proof.* The inclusion follows by Lemma 5. Consider the $\omega\omega$-SM $M_2$ shown in Figure 3. It maps the string $1^{\omega\omega}$ to $0^{\omega\omega}$; otherwise it is an identity. Clearly, it is complete and single-valued and thus is a GCA. Clearly, $G_{M_2}$ is not the global function of any CA. ∎

**Definition 2.** Let $M_i = (K^i, S, K^i_L, K^i_R, \gamma^i)$, $i = 1, 2$ be $\omega\omega$-SM. The *union* of $M_1$ and $M_2$ is denoted by $M_1 + M_2$. Assuming $K^1 \cap K^2 = \emptyset$, we define $M_1 + M_2 = (K^1 \cup K^2, S, K^1_L \cup K^2_L, K^1_R \cup K^2_R, \gamma^1 \cup \gamma^2)$. Clearly, $\rho(M_1 + M_2) = \rho(M_1) \cup \rho(M_2)$.

**Lemma 6.** *Let $M_1, M_2, \ldots, M_n$ be single-valued $\omega\omega$-SM such that $\bigcup_{i=1}^n \mathrm{dom}(M_i) = S^Z$, and $\mathrm{dom}(M_i) \cap \mathrm{dom}(M_j) = \emptyset$, for $i \neq j$; that is, the domains of $M_1, \ldots, M_n$ give a partition of $S^Z$. Then the $\omega\omega$-SM $= M_1 + M_2 + \ldots + M_n$ is a GCA.*

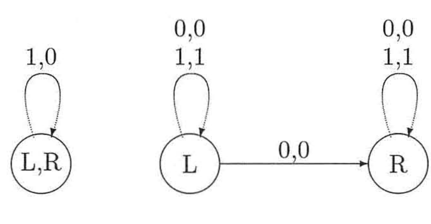



Figure 3: An $\omega\omega$-SM $M_2$.

*Proof.* The first property of the partition ensures that the union machine is complete, and the second one that it is single-valued. ■

**Example 3.** $\omega\omega$-SM $M_3$ shown in Figure 4 has the domain ${}^{\omega}0(0+1)^{*}.(0+1)^{\omega}$. Using the binary representation of rational numbers with the decimal point dividing the integer and the fractional part and with an infinite number of leading zeros, $M_3$ multiplies (syntactically correct) input by 3 and produces the output in the same notation (in $\rho(M_1)$ the decimal point remains in the same position). We use • to show the decimal point in the diagrams.

$\omega\omega$-SM $M_4$ shown in Figure 4 accepts any input with exactly one decimal point and infinite number of 1s to the left of the decimal point. It always produces ${}^{\omega}1.0^{\omega}$ as a representation of $\infty$.

$\omega\omega$-SM $M_5$ shown in Figure 4 accepts any input with less or more than one decimal point and produces ${}^{\omega}.^{\omega}$ (an error message).

Clearly, $M_3 + M_4 + M_5$ is a GCA.

**Corollary 2.** *Let $R_1, \ldots, R_n \subseteq S^{\omega\omega}$ be pairwise disjoint $\omega\omega$-regular sets, and let $M_1, \ldots, M_{n+1}$ be single-valued $\omega\omega$-SM such that $R_i \subseteq \mathrm{dom}^{\omega\omega}(M_i)$ for $i = 1, \ldots, n$ and $R_{n+1} = R_1 \cup \cdots \cup R_n \subseteq \mathrm{dom}^{\omega\omega}(M_{n+1})$. Let $\hat{M}_i$ be the restriction of $M_i$ to $R_i$ for $i = 1, \ldots, n+1$. Then $\hat{M}_1 + \hat{M}_2 + \ldots + \hat{M}_{n+1}$ is a GCA.*

*Proof.* By Theorem 2, $R_{n+1}$ is an $\omega\omega$-regular set. Hence, the result follows by Lemma 6. ■

Since by Lemma 5 every CA can be implemented by a single-valued $\omega\omega$-SM, we can always combine several CA working on different domains into one GCA.

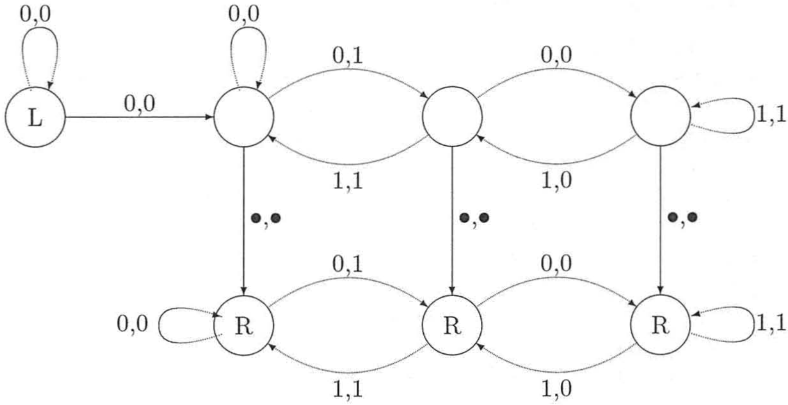
There are other ways to combine two or more CA into one GCA. Consider, for example, two CA $A_1$ and $A_2$ over alphabet (states) $\{a, b\}$. Then we can easily implement a GCA $M$ over alphabet $\{0, 1, a, b\}$ that preserves 0 and 1, simulates CA $A_1$ on every subconfiguration in $\{a, b\}^*$ between two 0s (with the neighborhood extended, e.g., cyclically), and simulates CA $A_2$ on the other subconfigurations in $\{a, b\}^*$, that is, between 0 and 1, 1 and 0, or 1 and 1.

## 4. Decision problems

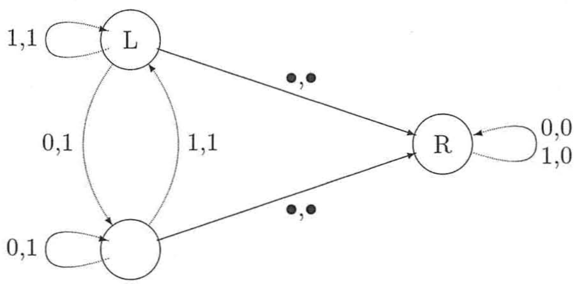
We considered some decision problems about $\omega\omega$-SM in section 3. Now we will study decision problems about GCA. Clearly, every problem that is undecidable for one-dimensional CA is also undecidable for GCA. So we will consider only those problems that are decidable for CA.

The (one-step) equivalence problem for global CA functions is trivially decidable; they are equivalent if and only if they are identical. This is not the case for GCA; however, the problem is still decidable.

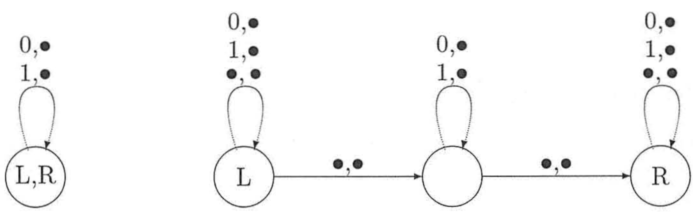
**Theorem 8.** *The equivalence problem for GCA is decidable.*




$\omega\omega$-SM $M_3$




$\omega\omega$-SM $M_4$




$\omega\omega$-SM $M_5$

Figure 4: GCA $M_3 + M_4 + M_5$.

*Proof.* Given two GCA, that is, two complete and single-valued $\omega\omega$-SM $M$ and $N$, we want to test whether $G_M(c) = G_N(c)$ for all $c \in S^Z$. Note that the equivalence of $M$ and $N$ on $S^{\omega\omega}$ does not imply the equivalence of $G_M$ and $G_N$ on $S^Z$; for example, the identity and shift are equivalent on $S^{\omega\omega}$ but not on $S^Z$. To prevent shifting we will replace $M$ and $N$ by (one-way) $\omega$-SM $\hat{M}$ and $\hat{N}$, respectively, as in the proof of Lemma 4. Clearly, $G_M \equiv G_N$ on $S^Z$ if and only if $\hat{M} \equiv \hat{N}$ on $(S \times S)^\omega$. The equivalence problem of single-valued $\omega$-transducer is shown to be decidable in [8]; $\hat{M}$ and $\hat{N}$ are (length-preserving) $\omega$-transducers. ■

Injectivity and surjectivity are well-known decidable problems for one-dimensional CA (see [1, 4]); we will extend these results to GCA. Note that because of the shift invariance, the injectivity of GCA $M$ on $S^Z$ clearly implies the injectivity of $G_M$ on $S^{\omega\omega}$. The converse is less obvious but holds, too. The case $k \geq 1, c \neq \sigma^k(c), G_M(c) = G_M(\sigma^k(c)) = d$ seems to violate the injectivity on $S^Z$ but not on $S^{\omega\omega}$. However, since $G_M$ is shift-invariant, we have $(\sigma^k(c), \sigma^k(d)) \in \rho(M)$, and the single-valuedness of $M$ implies that $\sigma^k(d) = d$. Thus, $G_M$ on $S^{\omega\omega}$ maps a string that does not have a period of length $k$ to one that does and therefore cannot be injective.

For GCA neither injectivity implies surjectivity nor does surjectivity imply injectivity. To show the former, consider $G$ defined on the $\omega\omega$-words in ${}^\omega 01^*0^\omega$ by the $\omega\omega$-SM $M_f$ from Figure 5 and otherwise as identity. Clearly, $G$ is injective and not surjective: $0^{\omega\omega}$ is not in the range of $G$. To show the latter, consider $G$ defined on strings from ${}^\omega 01^+0^\omega$ by the inverse (interchanged inputs and outputs) $M_f^{-1}$ of $\omega\omega$-SM $M_f$ from Figure 5, and again as identity elsewhere. Clearly, $M_f^{-1}$ is surjective but not injective since $G_{M_f}^{-1}({}^\omega 010^\omega) = G_{M_f}^{-1}({}^\omega 0^\omega) = {}^\omega 0^\omega$.

Following the terminology for one-dimensional CA (see e.g., [6]), we call a GCA $M$ *reversible* if an "inverse" GCA $N$ exists such that $G_M(c) = d$ if and only if $G_N(d) = c$ for all $c, d \in S^Z$.

The following theorems extend a well-known result for one-dimensional CA, (see, e.g., [6, Theorem 33]).

**Theorem 9.** *A GCA $M$ is reversible if and only if $G_M$ is bijective.*

*Proof.* Assume that $G_M$ is bijective. Let $M^{-1}$ be obtained by interchanging input and output symbols at every transition. Since $G_M$ is bijective, $M^{-1}$ is

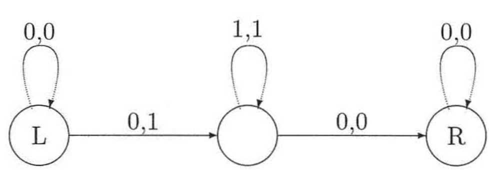



Figure 5: An $\omega\omega$-SM $M_f$.

single-valued and complete, so it is a GCA. Clearly, $M_{-1}$ defines the inverse of $G_M$.

The converse is obvious. ■

**Theorem 10.** *Given a GCA $M$, it is decidable whether $G_M$ is injective (on $S^Z$).*

*Proof.* Clearly, a GCA $M$ is injective if and only if its inverse $M^{-1}$ is single-valued. The latter is decidable by Lemma 4. ■

**Theorem 11.** *Given a GCA $M$, it is decidable whether $G_M$ is surjective (on $S^Z$).*

*Proof.* By Theorem 4 we can construct an $\omega\omega$-FA that accepts $G_M(S^Z)$. Clearly, $G_M$ is surjective if and only if $G_M(S^Z) = S^Z$, which is decidable by Theorem 5. ■

In the theory of CA, computation on (pseudo) finite configurations is frequently considered. Now we will study GCA working on (pseudo) finite configurations. Let us reserve 0 for the so-called quiescent symbol. A configuration $c$ in $S^Z$ is called *pseudo-finite* if there are $m, n$ such that $c = 0$ for all $i \leq m$ and all $i \geq n$. The $\omega\omega$-regular set ${}^\omega 0 S^\star 0^\omega$ is the set of bi-infinite words corresponding to (pseudo) finite configurations, We say that $\omega\omega$-SM $M = (K, S, K_L, K_R, \gamma)$ is a 0-GCA if $0 \in S$, $M$ is single-valued, $G_M$ is defined on all (pseudo) finite configurations (i.e., $\mathrm{dom}(\rho^{\omega\omega}(M)) \subseteq {}^\omega 0 S^\star 0^\omega$), and $G_M$ preserves (pseudo) finite configurations (i.e., $G_M^{\omega\omega}({}^\omega 0 S^\star 0^\omega) \subseteq {}^\omega 0 S^\star 0^\omega$).

**Lemma 7.** *It is decidable whether a given $\omega\omega$-SM is a 0-GCA.*

*Proof.* The proof follows from Theorems 2 and 4 and Lemma 4. ■

Using a simple modification of the proofs of Theorems 10 and 11, we have the following.

**Theorem 12.** *Given a 0-GCA $M$, it is decidable whether $M$ is (a) injective and (b) surjective on (pseudo) finite configurations.*

## 5. Limit sets

We can now extend to GCAs a number of results about the limit sets. First we review or introduce some notations. We present all results here in terms of bi-infinite sets of words that correspond to shift-invariant sets of configurations. Given a GCA $M$, we denote

$$\begin{aligned}\Omega_0 &= S^{\omega\omega},\\ \Omega_{k+1} &= G_M(\Omega_k) \qquad \text{for all } k \geq 0.\end{aligned}$$

Clearly, $\Omega_{k+1} \subseteq \Omega_k$ for all $k \in 0$. The *limit set* $\Omega_M$ of a GCA $M$ is the intersection of all forward images of $S^{\omega\omega}$, that is,

$$\Omega_M = \bigcap_{k=1}^{\infty} \Omega_k.$$

Clearly,

$$\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_K \supseteq \cdots \supseteq \Omega_M.$$

In terms of chaos theory, $\Omega_M$ is the attractor of the dynamical system specified by $G_M$.

For $C \subseteq S^{\omega\omega}$, we denote as $L(C)$ the set of all finite substrings of bi-infinite words in $C$, that is, $L(C) = \{w \in S^\star \mid u^R wv \in C \text{ for some } u, v \in S^\omega\}$. The following well-known result for one-dimensional CA can be extended to GCA.

**Theorem 13.** *For any GCA $M$ and every $k \geq 0$, $\Omega_k$ is an $\omega\omega$-regular set and $L(\Omega_k)$ is a regular set.*

*Proof.* Since $S^{\omega\omega}$ is $\omega\omega$-regular, the first part immediately follows from Theorem 4. Obviously, the substrings of any $\omega\omega$-regular set form a regular set. ■

Unlike the case for one-dimensional CA, the set $\Omega_k$ need not be closed; actually, we have the following.

**Lemma 8.** *Every $\omega\omega$-regular set $R$ that contains ${}^\omega a^\omega$ for some $a \in S$ can be obtained as $\Omega_1 = G_M(S^{\omega\omega})$ for some GCA $M$.*

*Proof.* It is easy to construct a GCA $M$ such that $G_M$ is the identity on $R$ and $G_M(w) = {}^\omega a^\omega$ for each $w$ in the complement of $R$, which is $\omega\omega$-regular by Theorem 2. ■

On the other hand, we have the following result, as for CA.

**Lemma 9.** *For each GCA $M$, for every $\Omega_k, k \geq 1$, and for $\Omega_M$ there exist $a_k, b \in S$ such that ${}^\omega a_k^\omega \in \Omega_k$ and ${}^\omega b^\omega \in \Omega_M$.*

*Proof.* Since $G_M$ is single-valued and shift-invariant, the image of any periodic bi-infinite word must be again periodic with the same period. So for each $a \in S$ there is $b \in S$ such that $G_M({}^\omega a^\omega) = {}^\omega b^\omega$, since ${}^\omega a^\omega \in S^{\omega\omega}$ ${}^\omega b^\omega \in \Omega_1$, and by induction there exists $a_k$ for each $k \geq 1$ such that ${}^\omega a_K^\omega \in \Omega_K$. Since $S$ is finite, there must be a cycle of length at most $|\, S \,|$ all of whose elements are in $\Omega_M$. ■

The limit language of GCA $M$ is the set of all finite substrings of the bi-infinite strings in the limit set $\Omega_M$. Another result (see [6, Theorem 24]) that clearly extends to GCAs is the following.

**Theorem 14.** *For every GCA $M$ the complement of the limit language, $S^\star - L(\Omega_M)$, is recursively enumerable.*

The limit language itself might not be recursively enumerable (see [6, Corollary 6]). However, unlike the case for one-dimensional CA, we have the following.

**Corollary 3.** *If for some GCA $M$, the limit set $\Omega_M$ is $\omega\omega$-regular, then there exists another GCA $N$ such that $\Omega_M = G_N(S^Z)$; that is, it is produced in one step by $N$.*

*Proof.* By Lemma 9 there is $b \in S$ such that ${}^\omega b^\omega \in \Omega_M$. Hence, the claim follows by Lemma 8. ■

## 6. Tilings and $\omega\omega$-sequential machines

It is well known that tilings of the infinite plane are closely related to the computation of CA. Here, following [5, 7, 12], we give an exact characterization of Wang tilings of the infinite plane in terms of the computation of $\omega\omega$-SAs. There is actually a one-to-one correspondence between the set of Wang tiles and simple $\omega\omega$-SMs.

We say that $\omega\omega$-SM $M = (K, S, \gamma, K_L, K_R)$ is simple if and only if $K_L = K_R = K$, that is, if all the states are left and right states. We can specify a simple $\omega\omega$-SM as a triple $M = (K, S, \gamma)$ and indicate no $L, R$ in its diagram.

*Wang tiles* are unit square tiles with colored edges. A tile whose left, right, top, and bottom edges have colors $p, q, r$, and $s$, respectively, is denoted by the 4-tuple $(p, q, r, s)$. A *tile set* is a finite set of Wang tiles. *Tilings* of the infinite euclidean plane are considered using arbitrarily many copies of the tiles in the given tile set. The tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiles may not be rotated. A tiling is *valid* if everywhere the contiguous edges have the same color.

Let $T$ be a finite tile set, and $f : \mathbb{Z}^2 \to T$ a tiling. Tiling $f$ is *periodic* with period $(a, b) \in \mathbb{Z}^2 - \{(0, 0)\}$ if and only if $f(x, y) = f(x + a, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. If there exists a periodic valid tiling with tiles of $T$, then there exists a *doubly periodic* valid tiling, that is, a tiling $f$ such that, for some $a, b > 0$, $f(x, y) = f(x + a, y) = f(x, y + b)$ for all $(x, y) \in \mathbb{Z}^2$. A tile set $T$ is called *aperiodic* if and only if (1) a valid tiling exists, and (2) no periodic valid tilings exist.

There is a one-to-one correspondence between the tile sets and sequential machines. This translates the properties of tile sets to the properties of computations of sequential machines. A finite tile set $T$ over a set of colors $C_{\mathrm{EW}}$ on east-west edges and a set of colors $C_{\mathrm{NS}}$ on north-south edges is represented by a sequential machine $M = (C_{\mathrm{EW}}, C_{\mathrm{NS}}, \gamma)$, where $(s, a, b, t) \in \gamma$ if and only if there is a tile $(s, a, b, t)$ in $T$; see Figure 6.

Obviously, bi-infinite words $x$ and $y$ are in the relation $\rho(M)$ if and only if there exists a row of tiles, with matching vertical edges, whose upper edges form sequence $x$ and lower edges sequence $y$. So there is a one-to-one correspondence between valid tilings of the plane and bi-infinite iterations of the sequential machine on bi-infinite sequences.

The two conditions for $T$ being aperiodic can be translated to conditions on computations of $M$. Clearly, set $T$ is aperiodic if (1) there exists a bi-

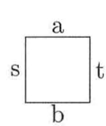



Figure 6: The tile $(s, a, b, t)$ corresponding to the transition $s \xrightarrow{a,b} t$.

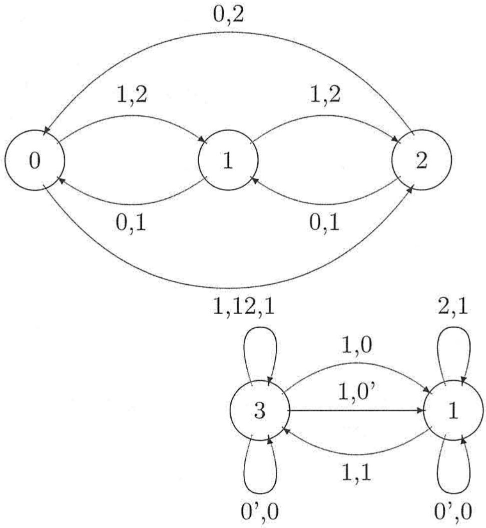



Figure 7: $\omega\omega$-SM $M_{13}$.

infinite computation of $M$ and (2) there is no bi-infinite word $w$ over $C_{\mathrm{NS}}$ such that $(w, w) \in [\rho(M)]^+$, where $\rho^+$ denotes the transitive closure of $\rho$.

In [5] it is shown that the simple $\omega\omega$-SM $M_{13}$ depicted in Figure 7 corresponds to an aperiodic set of tiles. This set consists of 13 tiles, corresponding to the edges of $M_{13}$, and it is the smallest aperiodic set known.

Note that if a simple $\omega\omega$-SM is a GCA, then for the reasons discussed in section 5, the corresponding set of Wang tiles allows rather simple periodic tilings.

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