Complex Systems 8 (1994) 161–180

Newton's Method for Quadratics, and Nested Intervals

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Abstract. The function $f_1(z) = \frac{1}{2}(z - 1/z)$ is the Newton map associated with the quadratic polynomial z^2+1. The iterated dynamics of f on $\Re$ is usually studied by exploiting the conjugacy between f and the function $z \mapsto z^2$. In this paper we show how to use the technique of *nested intervals* to yield a direct analysis of the dynamics. We also show how to use the same technique to analyze the dynamics of the function $f_2(z) = z - 1/z$.

1. Introduction

In this note we will show how to utilize the technique of *nested intervals* to examine the iterated dynamics of two related functions that map $\Re$, the set of real numbers, to itself. The two functions are

$$f_1(z) = \frac{1}{2}(z - 1/z) \tag{1}$$

and

$$f_2(z) = z - 1/z. \tag{2}$$

By studying the iterated dynamics of a function f we mean studying the infinite sequences generated by iterating the function f on initial seeds x_0, that is,

$$x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots, x_i = f(x_{i-1}), \ldots, \qquad i = 1, 2, \ldots \tag{3}$$

This sequence is sometimes known as the forward orbit of x_0. For $f = f_1$ this sequence has already been extensively studied because f_1 is the Newton map used to find the roots of the quadratic function $g(z) = z^2 + 1$. That is, given an initial seed x_0, the sequence (3) will usually converge to either i or

$-i$, the two roots of g. The set containing all x_0 for which the sequence (3) does not converge is the *Julia set* associated with f. The Julia set associated with both f_1 and f_2 is the real line $\Re$.

The usual method of analyzing the iterated dynamics of (1) uses the fact that f_1, considered as a mapping on the Riemann sphere, is conjugate under a linear fractional transformation to the function $z \mapsto z^2$. The purpose of this note is to illustrate how to analyze the dynamics directly without using the conjugacy relationship. The only tools we use are simple ones from elementary calculus and point-set topology. We exhibit a one-to-one correspondence between points in $\Re$ and integer sequences of alternating sign, for example, $5, -7, 4, -6, \ldots$. This correspondence has some nice properties. For example, if the sequence associated with x_0 is periodic/bounded; then (3) is also periodic/bounded. Furthermore, if the associated sequences of a series of points converges to the sequence associated with x, then the points themselves converge to x. We will use these properties to analyze the dynamics of f_1. We will also show how this analysis, slightly modified, can be used to analyze the dynamics of f_2.

In section 2 we quickly review Newton's method for finding the roots of polynomials. We also briefly sketch the conjugacy mapping that forms the basis for the usual analysis of f_1. In section 3 we explain what we mean by *nested intervals.* We then show how to use nested intervals to define the *behavior* of a point under f_1 in such a way that there is a one-to-one correspondence between points in $\Re$ and possible behaviors. This correspondence will yield immediate proofs of the standard facts about the iterated behavior of f. As an example, it will show that the preimages of any point in $\Re$ are dense in $\Re$. It will also show that the set of points that have bounded forward orbits is 'Cantor-like.' Finally, in section 4 we will show how to modify the analysis of section 3 so that it can be applied to the iterated behavior of f_2.

2. Newton's method

In this section we provide a quick review of Newton's method for finding the roots of a polynomial. For a more complete explanation see [4]. Let $g(z) = \sum_{i=0}^{n} a_i z^i$ be an nth-degree polynomial. The Newton's map associated with g is

$$f(z) = z - \frac{g(z)}{g'(z)}. \tag{4}$$

Newton's method for finding a root of g is to choose an initial seed x_0 and iterate f on x_0 to construct the infinite sequence (3). Later we will need a more flexible notation: we set $f^{(0)}(x) = x$ and inductively define $f^{(i)}(x) = f(f^{(i-1)}(x))$ for $i \geq 1$. In this new notation (3) is written as

$$x_0, f(x_0), f^{(2)}(x_0), \ldots, f^{(i)}(x_0), \ldots. \tag{5}$$

It is known that if x_0 is close enough to a root α of g, then the sequence x_i will converge to α. The set containing all points $x_0 \in \Re$ for which this sequence doesn't converge to some root of g is the Julia set associated with f [3].

As an example, suppose that $g(z) = (z-\alpha)^2$ where $\alpha \in \mathcal{C}$ is an arbitrary complex number. Then

$$f(z) = \frac{z+\alpha}{2}. \tag{6}$$

Therefore, for all $z \in \mathcal{C}$,

$$|\alpha - f(z)| = \frac{|\alpha - z|}{2}, \tag{7}$$

the sequence $x_i \rightarrow \alpha$ irrespective of the original seed x_0, and the Julia set associated with f is empty.

If g is a quadratic with two distinct roots the situation is more interesting. Suppose, for example, that $g(z) = z^2 + 1 = (z-i)(z+i)$. Then

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{8}$$

The classical method for analyzing f uses the fact that it is *conjugate* to the function $h(z) = z^2$. By conjugate we mean that there is some function T with a two-sided inverse T^{-1} such that

$$f(z) = T^{-1} \circ h \circ T(z) \tag{9}$$

where $\circ$ is the functional composition operator. By a two-sided inverse we mean that $T(T^{-1}(z)) = T^{-1}(T(z)) = z$. It is straightforward to check that equation (9) is true when $T(z) = (z+i)/(z-i)$. This T has the two-sided inverse $T^{-1}(z) = i(z+1)/(z-1)$.

Iterating (9) yields

$$x_i = T^{-1} \circ h^{(i)} \circ T(z), \tag{10}$$

which may be proved by induction. Therefore, the behavior of the sequence (5) can be examined by studying the behavior of the sequence $z, z^2, z^4, z^8, \ldots, z^{2^i}, \ldots$.

First let $S^1 = \{z : |z| = 1\}$ be the unit circle, $H^+ = \{x + iy : x, y \in \Re, y > 0\}$ be the half-plane with positive imaginary part, and $H^- = \{x - iy : x, y \in \Re, y > 0\}$ be the half-plane with negative imaginary part. The behavior of T is summarized in the following table.

z	i	$-i$	∞	$\Re$	H^+	H^-
$T(z)$	∞	0	1	S^1	$\{z : \|z\| > 1\}$	$\{z : \|z\| < 1\}$

If $|z| < 1$, then z^{2^i} converges to 0. If $|z| > 1$ then z^{2^i} goes to infinity. If $z \in S^1$, the unit circle, then every point in z^{2^i} is also on S^1. If $x_0 \in H^+$,

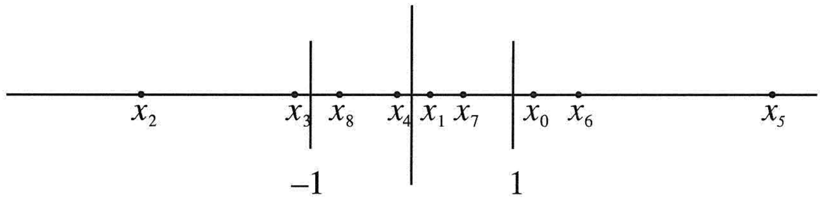

Figure 1: This figure illustrates the first nine iterates, $x_i = f^{(i)}(x_0)$, $0 \le i \le 8$ of f. Notice how the the sequence flip-flops between increasing/decreasing subsequences of positive/negative values.

then $z = |T(x_0)| > 1$, z^{2^i} tends to infinity, and $f^{(i)}(x_0) \to i$. Similarly, if $x_0 \in H^-$, then $z = |T(x_0)| < 1$, z^{2^i} tends to 0, and $f^{(i)}(x_0) \to -i$. Finally, if $x_0 \in \Re$, then $z = T(x_0) \in S^1$ and z^{2^i} is contained in S^1. Therefore the behavior of the sequence $f^{(i)}(x_0)$ can be studied by examining the behavior of the sequence z^{2^i} for $z \in S^1$.

In general, if $g(z) = (z-\alpha)(z-\beta)$ is a quadratic with two roots, then the analysis given above can be adapted to examine its associated Newton's function $f(z) = z - g(z)/g'(z)$ [4]. For more information on conjugacy methods and their applications to the study of iterated dynamics see [2].

In the next section we will show how to examine the dynamics of (5) without using the conjugacy relationship.

3. Nested intervals

A sequence of closed bounded intervals, $D_i = [a_i, b_i] \subseteq \Re$, $i = 1, 2, 3, \ldots$ is said to be *nested* if $D_i \subseteq D_{i-1}$ for $i > 1$. An important fact about such sequences of intervals, one which often occurs in the analysis of dynamical systems [5], is that the intersection $\bigcap_i D_i$ is a nonempty closed interval, possibly a single point. In this section we show how to utilize the technique of nested intervals to analyze the dynamics of (1). That is, for $x \in \Re$ we will study the behavior of (5).

Examination of f shows that it is a monotonically increasing surjective function from $[1, \infty)$ onto $[0, \infty)$ and from $(0, 1]$ to $(-\infty, 0]$. Similarly, it is a monotonically increasing surjective function from $(-\infty, -1]$ onto $(-\infty, 0]$ and from $[-1, 0)$ onto $[0, \infty)$. Therefore, the infinite sequence $f^{(i)}(x)$ will forever alternate between positive decreasing and negative increasing sequences (Figure 1) unless, for some i, $f^{(i)}(x) = 0$. In this case we say that the sequence terminates since $f(0)$ is undefined. (Alternatively we can say that $f(0) = f(\infty) = \infty$ and $f^{(j)}(x) = \infty$ for $j > i$.)

The rest of this section will be given over to showing that there is a one-to-one correspondence between points in $\Re$ and integer sequences of alternating sign

$$i_0, i_1, i_2, i_3, \ldots \qquad \text{such that } i_j i_{j+1} \le 0, \qquad \text{for } j \ge 0$$

for example, $1, -3, 5, -7, \ldots$ or $-4, 6, -7, 5, \ldots$. This correspondence possesses properties useful in the analysis of the iterated sequence (5). For example, (5) is bounded if and only if the alternating integer sequence associated with x is bounded.

We associate with each point $x \in \Re$ two sets: F_x and B_x. The set F_x, the forward orbit of x, is the set containing all of the iterates $f^{(i)}(x)$, while B_x, the backward orbit of x, is the set of all preimages of x. Formally

$$F_x = \{f^{(i)}(x) : i = 0, 1, 2, \ldots\}$$
$$B_x = \{z : \exists i, f^{(i)}(z) = x\}.$$

A technical note: If z is a preimage of 0 ($z \in B_0$), then there is some i such that $f^{(i)}(z) = 0$. In this case $f^{(i+1)}(z)$ is undefined and we define F_z to be the finite set $\{f^{(j)}(z) : 0 \leq j \leq i\}$.

Definition 1. *A point $x \in \Re$ is* bounded *if the forward orbit of x is bounded, that is, if there is some constant k such that $F_x \subset [-k, k]$.*

Definition 2. *A point $x \in \Re$ is* periodic *if there is some integer $i \geq 1$ such that $f^{(i)}(x) = x$.*

In this section we will prove the following facts:

Fact 1. $\forall x \in \Re$, the set B_x is dense in $\Re$.

Fact 2. The set $\{x : x \text{ is periodic}\}$ is dense in $\Re$.

Fact 3. The set $\{x : F_x \text{ is dense in } \Re\}$ is itself dense in $\Re$ and has cardinality $\aleph_1$.

Fact 4. The set $\{x : x \text{ is bounded}\}$ is dense in $\Re$, has cardinality $\aleph_1$ and measure 0. In fact, it is the union of a countable number of "Cantor-like" sets.

Notice that Fact 1 implies that the system that we are studying is chaotic. That is, it tells us that specifying a point x to a very high degree of precision is not enough to tell us how x behaves under iteration.

We define a set of intervals that partition the real line. First, for any given x the equation $f(z) = x$ has exactly two solutions: $z_\pm = x \pm \sqrt{x^2+1}$. Furthermore $f(-1/z) = f(z)$, so $z_- = -1/z_+$. Thus one of $z_\pm$ has absolute value greater than or equal to 1 while the other has the opposite sign and absolute value less than or equal to 1. We will denote these two solutions by $g(x)$ and $h(x)$. These functions are defined in such a way that (for $x \neq 0$) $g(-x) = -g(x)$ and $h(-x) = -h(x)$:

$$g(x) = \begin{cases} x + \sqrt{x^2+1}, & x \geq 0 \\ x - \sqrt{x^2+1}, & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} x - \sqrt{x^2+1}, & x \geq 0 \\ x + \sqrt{x^2+1}, & x < 0 \end{cases}.$$

Notice that g and h are both monotonically increasing functions in the intervals $(-\infty, 0)$ and $(0, \infty)$. Thus, if D is a bounded interval not containing

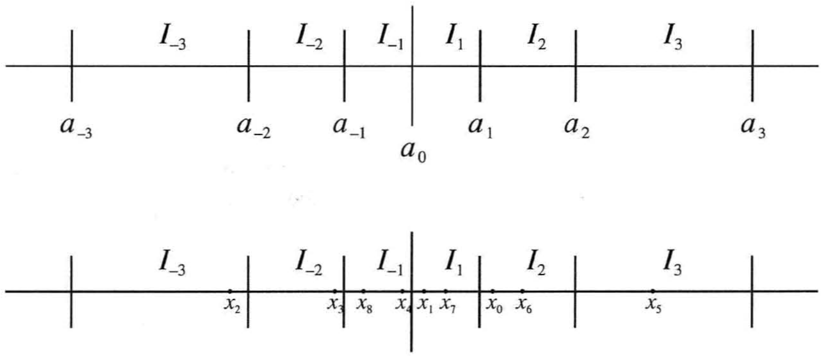

Figure 2: The top figure illustrates the location of the a_i: $a_0 = 0$, $a_{i+1} = g(a_i)$, $i > 0$ and $a_{-i} = -a_i$. The intervals I_i $(i > 0)$ are defined as $I_i = (a_{i-1}, a_i]$ with $I_{-i} = -I_i$. The bottom figure illustrates the first eight applications of f on the point $x = x_0$. We have set $x_i = f^{(i)}(x)$. The behavior of the sequence associated with x is $2, -3, 3, -1, \ldots$.

0, then both $g(D)$ and $h(D)$ will be bounded intervals that do not contain 0. These facts will be important later.

Next, we define a doubly infinite sequence a_i as follows.

$$a_0 = 0, \qquad a_i = g(a_{i-1}), \quad i > 0, \qquad a_i = -a_{-i}, \quad i < 0.$$

This permits us to define a set of intervals that partition $\Re$. Set $I_0 = [0, 0]$ and

$$I_i = \begin{cases} (a_{i-1}, a_i], & i > 0 \\ [a_i, a_{i+1}), & i < 0 \end{cases}. \tag{11}$$

Note that $I_i = -I_{-i}$ (see Figure 2). The functions f, g, and h operate rather remarkably on the intervals I_i. Recall that f is a monotonically increasing function from $[1, \infty)$ onto $[0, \infty)$ and $f(a_i) = a_{i-1}$ for $i \geq 1$. It follows that $f(I_i) = I_{i-1}$ for $i > 1$. (We use the notation $f(S) = \{f(x) : x \in S\}$ where S is an arbitrary set.) Symmetrically, $f(I_i) = I_{i+1}$ for $i < 1$. There are similar results for g and h. We summarize them in the following table:

i	$=1$	$=-1$	>1	<-1
$f(I_i) =$	$(-\infty, 0)$	$(0, \infty)$	I_{i-1}	I_{i+1}
$g(I_i) =$	I_2	I_{-2}	I_{i+1}	I_{i-i}
$h(I_i) \subseteq$	I_{-1}	I_1	I_{-1}	I_1

(12)

We can associate with every $x \in \Re$ a unique interval I_i that contains it (because the I_i form a partition of $\Re$) and thus with the iterated sequence

$$x, f(x), f^{(2)}(x), \ldots, f^{(j)}(x), \ldots$$

we can associate a unique sequence of integers

$$i_0, i_1, i_2, \ldots, i_j, \ldots$$

such that $f^{(j)}(x) \in I_{i_j}$. As an example the point $x = x_0$ pictured in Figure 2 has the associated sequence

$$2, 1, -3, -2, -1, 3, 2, 1, -1, \ldots \tag{13}$$

because $x \in I_2$, $f(x) \in I_1$, $f^{(2)}(x) \in I_{-3}$, and so forth. If $x \in B_0$, then its associated integer sequence is finite and ends at 0 since $I_0 = \{0\}$ and $f(0)$ is undefined. For example, if $x = g(g(h(a_5)))$, then its associated sequence is

$$-3, -2, -1, 5, 4, 3, 2, 1, 0. \tag{14}$$

If $x \notin B_0$, then the sequence associated with x is infinite.

Notice that both (13) and (14) share a peculiar property: they are composed of concatenations of positive subsequences that step down by one to 1 and negative subsequences that step up by one to -1. This property is a direct result of the first row of (12). More specifically, if $x \in I_i$, $i > 1$, then $f(x) \in I_{i-1}, \ldots, f^{(i-1)}(x) \in I_1$ and $f^{(i)}(x) \in I_j$ where $j \leq 0$. Symmetrically, if $x \in I_i$, $i < -1$, then $f(x) \in I_{i+1}, \ldots, f^{(|i|-1)}(x) \in I_{-1}$ and $f^{(|i|)}(x) \in I_j$ where $j \geq 0$. The process terminates if and only if $i = 0$, since then $x = 0$ and $f(0)$ is undefined.

What we have just described is a more detailed description of the "flip-flopping" behavior illustrated in Figure 1. In the next few pages (culminating in Theorem 1) we will show that there is a one-to-one correspondence between sequences of flip-flops and points in $\Re$. We will then use this correspondence to prove Facts 1, 2, 3, and 4.

Thus, given $x \in I_i$, the first iterate of f on x whose location is unknown is $f^{(|i|)}(x)$. The sequence associated with x can therefore be reconstructed from a sequence containing only the first elements of the increasing (decreasing) sequences, that is, $2, -3, 3, -1, \ldots$ in place of (13) and $-3, 5, 0$ in place of (14). We will call this abbreviated sequence the *behavior* of the sequence associated with x, or simply the behavior of x. We denote the behavior by S_x. The formal definition of S_x follows.

Definition 3. *For a point $x \in \Re$ its* behavior, *S_x, is the sequence defined recursively as follows. Let i be the index of the interval containing x, that is, $x \in I_i$.*

1. *If $i = 0$, then S_x is the one item sequence 0.*
2. *If $i \neq 0$, then $S_x = i, S_{f^{(|i|)}(x)}$.*

For a given point x we will denote S_x by

$$S_x = i_0^x, i_1^x, i_2^x, \ldots.$$

From the definition we see that

$$f^{(|i_0^x|+|i_1^x|+\cdots+|i_{n-1}^x|)}(x) \in I_{i_n}.$$

Definition 4. *A sequence of nonzero integers $i_0, i_1, i_2, \ldots$ (finite or infinite)* alternates *if* $\operatorname{sign}(i_j) = -\operatorname{sign}(i_{j+1})$, $j = 0, 1, 2 \ldots$.

Definition 5. *An integer sequence S that satisfies one of the following two conditions will be called a* legal behavior.

1. *S is an infinite alternating sequence; or*
2. *$S = S', 0$ where S' is a finite alternating sequence.*

For example, $S = 1, -1, 1, 0$ and $1, -2, 3, -4, 5, -6, \ldots$ are legal behaviors, whereas $1, -1$ and $1, 1, 1, 1, 1, \ldots$ are not. We will use the notation $S = i_0, i_1, i_2, i_3, \ldots$ to represent the component integers of S. Thus the behavior $S = 1, -2, 3, -4, 5, -6, \ldots$ can be expressed by writing $i_k = (-1)^k(k+1)$, $k \geq 0$.

Definition 6. *Let $i_0, i_1, \ldots, i_n$ be a finite alternating sequence. Then*

$$D(i_0, i_1, \ldots, i_n) = \{x : i_k^x = i_k, \quad 0 \leq k \leq n\}$$

is the set of all x such that the first $n+1$ components of S_x are identical with $i_0, i_1, \ldots, i_n$.

For example, $D(2, -3, 3)$ contains all points $x \in I_2$ such that $f^{(2)}(x) \in I_{-3}$ and $f^{(2+3)} \in I_3$. This is illustrated in Figure 3. The set $D(i_0, i_1, \ldots, i_n)$ can be constructed explicitly as follows. First note that $D(i_0) = \{x : x \in I_{i_0}\} = I_{i_0}$. We also have the following lemma.

Lemma 1. *Suppose $i_0, i_1, \ldots, i_n$ is an alternating sequence. Then*

$$g(D(i_0, i_1, \ldots, i_n)) = \begin{cases} D(i_0 + 1, i_2, \ldots, i_n), & i_0 > 0 \\ D(i_0 - 1, i_2, \ldots, i_n), & i_0 < 0 \end{cases} \tag{15}$$

and

$$h(D(i_0, i_1, \ldots, i_n)) = \begin{cases} D(-1, i_0, i_2, \ldots, i_n), & i_0 > 0 \\ D(1, i_0, i_2, \ldots, i_n), & i_0 < 0. \end{cases} \tag{16}$$

An immediate application of Lemma 1 is that for $i_0, i_1, \ldots, i_n$, an alternating sequence

$$D(i_0, i_1, \ldots, i_n) = g^{(|i_0|-1)}(h(D(i_1, i_2, \ldots, i_n))). \tag{17}$$

As an example, we show how to apply the lemma to construct the intervals illustrated in Figure 3.

1. $D(2) = I_2$;
2. $D(-3) = I_{-3}$ so $D(2, -3) = g(h(I_{-3}))$;
3. $D(3) = I_3$ so $D(-3, 3) = g^{(2)}(h(I_3))$ and $D(2, -3, 3) = g(h(D(-3, 3)))$ $= g(h(g^{(2)}(h(I_3))))$.

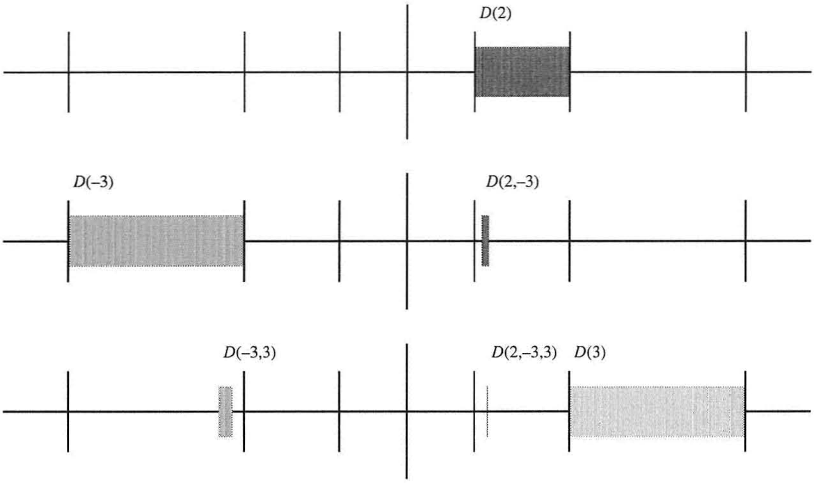

Figure 3: This figure exhibits how the nested intervals converge to a point. In the example we show the first three steps in the process of finding a point x with $S_x = 2, -3, 3, \ldots$. The top row shows $D(2)$, the middle row $D(2,-3)$, and the bottom row $D(2,-3,2)$.

In general, to explicitly construct $D(i_0, i_1, \ldots, i_n)$ we set

$$t_0(x) = x, \qquad t_n(x) = t_{n-1} \circ g^{(|i_{n-1}|-1)} \circ h(x). \tag{18}$$

Repeated application of (17) yields

$$D(i_0, i_1, \ldots, i_n) = t_n(D(i_n)) = t_n(I_{i_n}).$$

As mentioned before, if D is an interval not containing zero, then so are $h(D)$ and $g(D)$; thus $t_n(D)$ is an interval not containing zero. This proves that $D(i_0, i_1, \ldots, i_n)$ is an interval that does not contain zero. Furthermore, h and g are one-to-one functions so t_n is a one-to-one function that maps I_n onto $D(i_0, i_1, \ldots, i_n)$. This will be important in the proof of Theorem 1.

Another consequence of Lemma 1 is an upper bound on the size of $D(i_0, i_1, \ldots, i_n)$ that decreases geometrically with n. We use $\mu(D)$ to denote the standard Lebesgue measure of set D.

Lemma 2. *Set $r = 1 - (a_2^2 + 1)^{-1/2} \approx .6173$. For $i_0, i_1, \ldots, i_n$, an alternating sequence,*

$$\mu(D(i_0, i_1, \ldots, i_n)) \leq 2^{|i_0|} \cdot r^n.$$

Proof. Our main tool will be the following variant of the the mean value theorem: if D is an interval and s a continuously differentiable function, then

$$\mu(s(D)) = |s'(\xi)| \cdot \mu(D) \text{ for some } \xi \in D. \tag{19}$$

As a first application, notice that $h \circ g \circ f(x) = h(-1/x)$ for $0 < x < 1$. Since, for $x \neq 0$, we have $h'(x) = 1 - |x|/\sqrt{1+x^2}$, we immediately find that for all $0 < x < 1$

$$(h \circ g \circ f)'(x) = \frac{\sqrt{1+x^2}-1}{x^2\sqrt{1+x^2}} < \frac{\sqrt{1+x^2}-1}{x^2} \leq \frac{1}{2}.$$

Applying the mean value theorem gives that for any interval $D \subseteq (0,1)$,

$$\mu(h \circ g \circ f(D)) < \frac{1}{2}\mu(D). \tag{20}$$

To begin, let $i > 0$. Then from Lemma 1 and the equation $f(h(x)) = x$ we have

$$\begin{aligned} D(-1,i) &= h(I_i) \\ &= h(g^{(i-1)}(I_1)) \\ &= (h \circ g \circ f)^{(i-1)}(h(I_1)). \end{aligned}$$

An immediate consequence is that $\mu(D(-1,i)) \leq 2^{1-i}$. Since $D(-1,i) = -D(1,-i)$ we have just proven that for i_0, i_1 alternating, $i_0 = \pm 1$,

$$\mu(D(i_0,\, i_1)) \leq 2^{1-|i_1|}. \tag{21}$$

Next, let $i_0, i_1, \ldots, i_n$ be an alternating sequence with $i_0 = -1$. Again using Lemma 1 we can write

$$\begin{aligned} D(-1, i_1, \ldots, i_n) &= h \circ g^{(i_1-1)}(D(1, i_2, \ldots, i_n)) \\ &= (h \circ g \circ f)^{(i_1-1)}(h(D(1, i_2, \ldots, i_n))). \end{aligned}$$

Therefore,

$$\mu(D(-1, i_1, \ldots, i_n)) \leq 2^{1-i_1}\mu(h((D(1, i_2, \ldots, i_n)))). \tag{22}$$

But $h'(x) = 1 - (|x|/\sqrt{1+x^2})$ and

$$D(1, i_2, \ldots, i_n) \subseteq \left[\frac{1}{a_{1-i_2}}, \frac{1}{a_{-i_2}}\right],$$

so another application of the mean value theorem gives

$$\mu(h(D(1, i_2, \ldots, i_n))) \leq \left(1 - \frac{1}{\sqrt{a_{i_2+1}^2+1}}\right)\mu(D(1, i_2, \ldots, i_n)). \tag{23}$$

Substituting back into (22) and taking symmetry into account we have just proven that for $i_0, i_1, \ldots, i_n$, an alternating sequence with $i_0 = \pm 1$,

$$\mu(D(i_0, i_1, \ldots, i_n)) \leq \frac{\left(1 - (a_{|i_2|+1}^2 + 1)^{-1/2}\right)}{2^{|i_1|-1}}\mu\left(D\left(\frac{i_1}{|i_1|}, i_2, i_3, \ldots, i_n\right)\right). \tag{24}$$

We unravel this inequality by recursively applying it to its own right-hand side $n-1$ times and then apply (21) once, obtaining

$$\mu\left(D\left(\pm 1,\, i_1,\, i_2,\, \ldots,\, i_n\right)\right) \leq \prod_{2\leq j\leq n} 2^{1-|i_j|}\left(1-\frac{1}{\sqrt{a_{|i_j|+1}^2+1}}\right).$$

We now examine this product on a term-by-term basis. If $|i_j| = 1$, then $1-(a_{|i_j|+1}^2+1)^{-1/2} = r$, while if $|i_j| > 1$, then $2^{1-|i_j|} \leq 1/2 < r$. Therefore,

$$\mu\left(D\left(\pm 1, i_1, i_2, \ldots, i_n\right)\right) \leq r^{n-1}.$$

Until now we assumed that $i_0 = \pm 1$. We conclude the proof of the lemma by noting that, for arbitrary i_0,

$$D(i_0, i_1, \ldots, i_n) = g^{(i_0-1)}\left(D\left(1, i_1, i_2, \ldots, i_n\right)\right).$$

For all $x \neq 0$, we have $g'(x) = 1 + |x|/\sqrt{1+x^2} \leq 2$. The lemma therefore follows from another application of the mean value theorem. ■

Definition 7. *Let $S = i_0, i_1, i_2, \ldots$ be an infinite legal behavior. Let $S^k = i_0^k, i_1^k, i_2^k, \ldots$ be a sequence of (finite or infinite) behaviors. We say that S^k* converges *to S (and write $S^k \to S$) if the S^k converge to S component-wise, that is,*

$$\forall j,\ \exists N_j \quad \textit{such that} \quad \forall k > N_j,\ \forall j' \leq j, \quad i_{j'}^k = i_{j'}.$$

For example, let S be the sequence $S_j = (-1)^j(j+1)$ and S^k the sequences $S_j^k = (-1)^j\left((j \bmod k)+1\right)$. Then

$$\begin{aligned} S &= 1, -2, 3, -4, 5, -6, \ldots \\ S^1 &= 1, -1, 1, -1, 1, -1, 1, -1, \ldots \\ S^2 &= 1, -2, 1, -2, 1, -2, 1, -2, \ldots \\ S^3 &= 1, -2, 3, -1, 2, -3, 1, -2, \ldots \end{aligned}$$

and $S_k \to S$.

We now formulate and prove our main theorem.

Theorem 1. *The correspondence*

$$x \leftrightarrow S_x$$

is a one-to-one and onto mapping between $\Re$ and the set of legal behaviors. Furthermore, if x is a point and x_i is a sequence of points such that $S_{x_i} \to S_x$, then $x_i \to x$.

Proof. To prove the first part, we must show that for every legal behavior S there is a unique point x such that $S_x = S$. We will treat the two cases S finite and S infinite separately.

First assume that S is a finite alternating sequence

$$S = i_0, i_1, \ldots, i_n, 0.$$

Recall the function t_n defined in (18), a one-to-one function with the property

$$D(i_0, i_1, \ldots, i_n) = t_n(I_{i_n}).$$

We claim that $x = t_n(a_{i_n})$ is the unique point such that $S_x = S$. This x satisfies $S_x = S$ because $x \in D(i_0, i_1, \ldots, i_n)$ and $f^{(|i_n|)}(a_{i_n}) = 0$. Suppose now that $x \neq t_n(a_{i_n})$ is another point such that $S_x = S$. Then $x \in D(i_0, i_1, \ldots, i_n)$ so there is a unique $x' \in I_{i_n}$ such that $x = t(x')$. Furthermore, we must have $f^{(|i_n|)}(x') = 0$. The unique x' that satisfies this last condition is $x' = a_{i_n}$. Therefore $x = t_n(a_{i_n})$ is the *unique* point satisfying $S_x = S$.

Now assume that $S = i_0, i_1, \ldots, i_n, \ldots$ is infinite. By definition,

$$S_x = S \qquad \Longleftrightarrow \qquad x \in \bigcap_n D(i_0, i_1, \ldots, i_n).$$

To simplify our notation we set $D_n = D(i_0, i_1, \ldots, i_n)$. We have already seen that the D_n are nested intervals with $\mu(D_n) \downarrow 0$. If the intervals were also *closed*, then as mentioned in the first paragraph in this section, there would be a unique x such that $\cap_n D_n = \{x\}$ and we would be finished.

But the D_n are not closed; they are half-open intervals such as $D(5) = I_5 = (a_4, a_5]$. Thus $\overline{D_n}$, the closure of D_n, *is* closed, that is, $\overline{D(5)} = [a_4, a_5]$. Furthermore, $\mu(\overline{D_n}) = \mu(D_n) \downarrow 0$ so there is a unique x such that $\cap_n \overline{D_n} = \{x\}$. To prove that there is a unique x such that $S_x = S$, it therefore suffices to show that $\cap_n D_n = \cap_n \overline{D_n}$.

Recall that $D(i_0, i_1, \ldots, i_n) = t_n(I_{i_n})$ where

$$t_0(x) = x, \qquad t_n(x) = t_{n-1} \circ g^{(|i_{n-1}|-1)} \circ h(x).$$

The function t_n is the repeated composition of the functions h and g. Both of these functions are continuous except at $x = 0$ and map $\Re$ to $\Re \setminus \{0\}$. Thus

$$\begin{aligned}\overline{D(i_0, i_1, \ldots, i_n)} &= \overline{t_n(I_{i_n})} \\ &= \overline{t_{n-1} \circ g^{(|i_{n-1}|-1)} \circ h(I_{i_n})} \\ &= t_{n-1} \circ g^{(|i_{n-1}|-1)}(\overline{h(I_{i_n})}).\end{aligned}$$

The next step is to calculate $\overline{h(I_i)}$ for all $i \neq 0$. If $i > 0$, then $I_i = (a_{i-1}, a_i]$ and so $h(I_i) = \left(-\frac{1}{a_i}, -\frac{1}{a_{i+1}}\right]$. Thus $\overline{h(I_i)} = h(I_i) \cup \{-\frac{1}{a_i}\}$. A similar calculation shows that this remains true even when $i < 0$. Therefore we have shown that

$$\overline{D(i_0, i_1, \ldots, i_n)} = D(i_0, i_1, \ldots, i_n) \cup \{t_{n-1} \circ g^{(|i_{n-1}|-1)}(u_n)\}$$

where $u_n = -1/a_{i_n}$. We can rewrite this as $\overline{D_n} = D_n \cup \{x_n\}$ where $x_n = t_{n-1} \circ g^{(|i_{n-1}|-1)}(u_n)$. By definition we have

$$f^{(|i_0|+|i_1|+\cdots+|i_{n-2}|+|i_{n-1}|-1)}(x_n) = u_n \tag{25}$$

and thus $x_n \in B_0$, the set of preimages of 0.

Suppose now that $\{x\} = \cap_n \overline{D_n} \neq \cap_n D_n$. Because the D_n are nested there must be some integer N such that for all $n \geq N$, $x \in \overline{D_n}$; but $x \notin D_n$, so $x = x_n$ for all $n \geq N$. From what we have seen above we know that $x = x_N \in B_0$, so S_x is finite. Thus there is some m such that $f^{(m)}(x) = 0$; any further iterate of x will be undefined. In particular, $f^{(|i_0|+|i_1|+\cdots+|i_{m-1}|+|i_m|)}(x)$ will be undefined. But this contradicts (25) so we must have $\{x\} = \cap_n \overline{D_n} = \cap_n D_n$, and thus for every S, there is a unique point x with $S_x = S$.

We now prove that $S_{x_i} \to S_x$ implies $x_i \to x$. We must show that for every $\epsilon > 0$ there is an N such that for all $n > N$, $|x_n - x| \leq \epsilon$. This is straightforward. Given s let j be the first integer such that $\mu(D(i_0, i_1, \ldots, i_j)) \leq \epsilon$. Lemma 2 tells us that such a j must exist. Since $S_{x_i} \to S_x$ there must be an N such that for all $n > N$ and for all $k \leq j$, $i_k^{x_n} = i_k^x$. Therefore $x_n, x \in D(i_0, i_1, \ldots, i_j)$ and so $|x_n - x| \leq \epsilon$. ∎

The theorem lets us derive properties describing the iterated dynamics of f. We use the fact that if S_y is a *suffix* of S_x, then $y \in F_x$ and $x \in B_y$. By suffix we mean that there is some $n > 0$ such that $i_j^y = i_{n+j}^x$ for all $j \geq 0$. Theorem 1 tells us that for a given x and fixed n there is a unique y that fulfills this condition. The definition of S_x tells us that $y = f^{(|i_0|+|i_1|+\cdots+|i_{n-1}|)}(x)$. Thus $y \in F_x$ and $x \in B_y$.

As an example, suppose that

$$S_y = 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, -12, \ldots$$
$$S_x = 13, -13, 13, -13, 1, -2, 3, -4, 5, -6, 7, -8, \ldots.$$

Then S_y is a suffix of S_x.

Recall that a point is periodic if there is some j such that $f^{(j)}(x) = x$. The discussion in the previous paragraph implies that if S_x is periodic in the sense that there is some $n > 0$ such that $i_j^x = i_{j+n}^x$ for all $j \geq 0$, then x is periodic (the converse is almost but not quite true). Thus we can prove Fact 2.

Lemma 3. *Let $P = \{x : \exists j > 0$ such that $f^{(j)}(x) = x\}$ be the set of periodic points. Then P is dense in $\Re$.*

Proof. We will actually show that P is dense in the set of all points with infinite behaviors, $\Re \setminus B_0$. Since B_0 is countable the proof will follow. The general idea is to construct a sequence of periodic behaviors S_{x_n} that converge to S_x. For example, if

$$S_x = 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots,$$

then we might choose x_n so that

$$S_{x_1} = 1, -2, 1, -2, 1, -2, 1, -2, 1, -2, \dots$$
$$S_{x_2} = 1, -2, 3, -4, 1, -2, 3, -4, 1, -2, \dots$$
$$S_{x_3} = 1, -2, 3, -4, 5, -6, 1, -2, 3, -4, \dots$$

and so forth.

Formally let $x \in \Re \backslash B_0$ with $S_x = i_0^x, i_1^x, \dots$. Then by choice, S_x is infinite. For $n > 0$, let x_n be the unique point that satisfies $i_j^{x_n} = i^x_{j \bmod (2n)}$. Then

$$S_{x_n} = i_0^x, i_1^x, \dots, i_{2n-1}^x, i_0^x, i_1^x, \dots, i_{2n-1}^x, \dots.$$

(The modulus is taken $2n$ and not n to ensure that S_{x_n} alternates.) By definition, $x_n \in P$. It is not hard to see that $S_{x_n} \to S_x$, so $x_n \to x$ and we have finished the proof. ∎

We use the same technique to prove Fact 1.

Lemma 4. *For all $x \in \Re$, the set B_x is dense in $\Re$.*

Proof. Fix $x \in \Re$. The behavior $S_x = i_0^x, i_1^x, \dots$ can be finite or infinite. As in the previous lemma it will be enough to show that B_x is dense in $\Re \setminus B_0$.

Let y be an arbitrary point in $\Re \setminus B_0$, that is, $S_y = i_0^y, i_1^y, \dots$ is infinite. We construct a sequence of points x_n such that $S_{x_n} \to S_y$. Furthermore, S_x will be a suffix of each of the S_{x_n} so $x_n \in B_x$. The proof of the lemma will follow from Theorem 1. As an example suppose that

$$S_x = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \dots$$
$$S_y = 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \dots.$$

We can choose the x_n such that

$$S_{x_1} = 1, -2, 1, -1, 1, -1, 1, -1, 1-, 1, \dots$$
$$S_{x_2} = 1, -2, 3, -4, 1, -1, 1, -1, 1-, 1, \dots$$
$$S_{x_3} = 1, -2, 3, -4, 5, -6, 1, -1, 1-, 1, \dots$$
$$S_{x_4} = 1, -2, 1, -1, 5, -6, 7, -8, 1-, 1, \dots$$

Formally, we construct the x_n so that S_{x_n} starts out as S_y but ends as S_x. To do this we define a parameter δ that ensures that the copies of S_x commence at locations in the S_{x_n} that have the proper parity. Let

$$\delta = \begin{cases} 0 & \text{if } \operatorname{sign}(i_0^x) = \operatorname{sign}(i_0^y) \\ 1 & \text{if } \operatorname{sign}(i_0^x) \neq \operatorname{sign}(i_0^y) \end{cases}$$

where $\operatorname{sign}(i) = i/|i|$. We now set x_n to be the unique point such that

$$i_k^{x_n} = \begin{cases} i_k^y & 0 \le k \le 2n + \delta \\ i^x_{k-(2n+\delta+1)} & k \ge 2n + \delta + 1 \end{cases}.$$

The points x_n are all in B_x and $S_{x_n} \to S_y$; therefore $x_n \to y$ and we have finished the proof. ∎

Utilizing the same technique, we now prove Fact 3.

Lemma 5. *The set*

$$D = \{x : F_x \text{ is dense in } \Re\}$$

is itself dense in $\Re$ *and has cardinality* $\aleph_1$.

Proof. We start by showing how to construct a point $x \in D$. Let A be the set of all finite alternating sequences with odd length whose first element is positive. It includes sequences such as $\alpha = 1, -3, 5$. For any sequence α, we write $-\alpha$ to denote the sequence whose elements are the negatives of those in α. For the given example $-\alpha = -1, 3, -5$. Now A is a countable set so we can enumerate all the sequences in A as $\alpha_1, \alpha_2, \ldots$. Let S be the concatenation of all of the pairs $\alpha_i, -\alpha_i$. That is,

$$S = \alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_3, -\alpha_3, \ldots.$$

Let x be the unique point with $S_x = S$. We claim that F_x is dense in $\Re$.

Let $y \in \Re \setminus B_0$ and $\epsilon > 0$. We must show that there is some i such that $|f^{(i)}(x) - y| < \epsilon$. From Lemma 2 we know that there is some n such that if $y' \in D\,(i_0^y, i_1^y, \ldots, i_n^y)$, then $|y' - y| < \epsilon$. Now, by definition, if the sequence $i_0^y, i_1^y, \ldots, i_n^y$ appears anywhere in S_x, then there is some i such that $f^{(i)}(x) \in D\,(i_0^y, i_1^y, \ldots, i_n^y)$. By construction we know that *every* finite alternating sequence appears somewhere in S_x. Thus there is some i such that $|f^{(i)}(x) - y| < \epsilon$. Since y and ϵ were chosen arbitrarily we have just shown that F_x is dense in $\Re$.

It is easy to modify the construction to show that D has cardinality $\aleph_1$. For each $i = 1, 2, 3, \ldots$ choose β_i to be one of the two sequences $1, -1$ or $2, -2$. Let S be the concatenation of all of the triplets $\alpha_i, -\alpha_i, \beta_i$. That is

$$S = \alpha_1, -\alpha_1, \beta_1, \alpha_2, -\alpha_2, \beta_2, \alpha_3, -\alpha_3, \beta_3, \ldots.$$

Let x be the unique point with $S_x = S$. The analysis of the previous paragraph shows that F_x is dense in $\Re$. Since there are $\aleph_1$ possible choices of the sequences $\beta_1, \beta_2, \beta_3, \ldots$, there are at least $\aleph_1$ points x with $F_x \in D$.

It remains to be shown that D itself is dense in $\Re$. This is trivial. If $x \in D$, then $f(x) \in D$ so $F_x \subseteq D$. Since F_x is dense in $\Re$ so is D. ∎

We conclude this section by analyzing the structure of the set of all bounded points. This set will be shown to be the union of a countable number of sets, each possessing a structure similar to that of the Cantor set (Fact 4).

Theorem 2. *The set of bounded points*

$$S = \{x : \exists c > 0 \quad \text{such that} \quad F_x \subseteq [-c, c]\}$$

has cardinality $\aleph_1$ *and measure 0.*

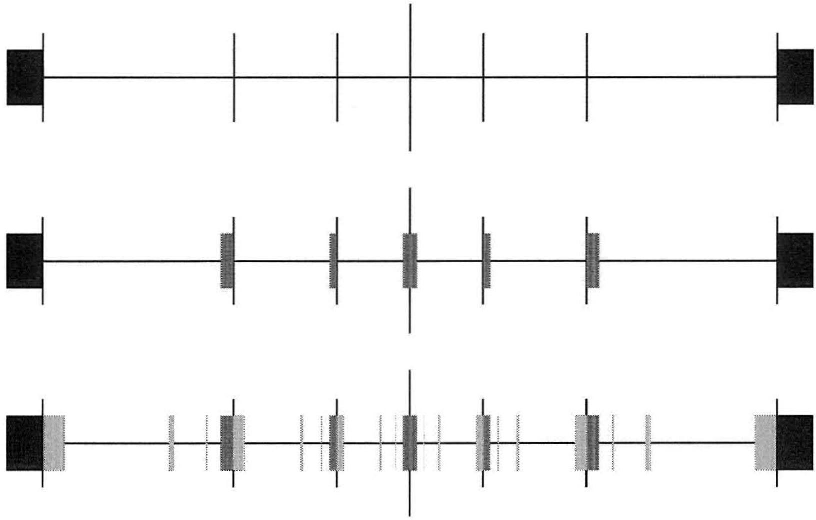

Figure 4: This figure illustrates the first three steps in the construction of S_3, the set of all points whose forward iterates are in the bounded interval $[-a_3, a_3]$. The unshaded area in the top diagram is D_0, in the middle D_1, and in the bottom D_2. At each step $D_{n+1} = D_n \setminus C_n$.

Proof. We define S_m, the set of points whose forward orbit is in $[-a_m, a_m]$:

$$S_m = \{x : F_x \subseteq [-a_m, a_m]\}. \tag{26}$$

Since $a_m \uparrow \infty$ we have $S_1 \subseteq S_2 \subseteq S_3 \ldots$ and $S = \cup S_n$. We will prove that the cardinality of S_2 is $\aleph_1$ and therefore so is the cardinality of S. We will also prove that for every m, $\mu(S_m) = 0$ and thus $\mu(S) = \mu(\cup_m S_m) = 0$ because the countable union of sets of measure 0 has measure 0.

Our main tool will again be the correspondence $x \leftrightarrow S_x$. As before, for $x \in \Re$ we denote

$$S_x = i_0^x, i_1^x, i_2^x, \ldots.$$

With this notation it is easy to see that (26) can be rewritten as

$$S_m = \{x : \forall j, |i_j^x| \leq m\}. \tag{27}$$

For example, $S_1 = \{x, -x\}$ where x is the unique point such that

$$S_x = 1, -1, 1, -1, 1, -1, \ldots.$$

In fact, $x = 1/\sqrt{3}$. (This point x can be found by solving $f(x) = -x$.)

That S_2 (and therefore S) has cardinality $\aleph_1$ follows from Theorem 1 together with the fact that the set of infinite alternating sequences that can be constructed utilizing the integers $1, -1, 2, -2$ has cardinality $\aleph_1$. The second part, that $\mu(S_m) = 0$, will be more difficult to prove.

For the rest of the proof we assume that $m > 1$ is fixed. We set

$$D_n = \bigcup_{\substack{i_0,i_1,\ldots,i_n \\ \forall j\le n, \quad |i_j|\le m}} D(i_0, i_1, \ldots, i_n)$$

where the union is taken over all alternating sequences of length $n+1$. With this definition $S_m = \bigcap_n D_n$, so it will be enough to show that $\mu(D_n) \to 0$.

In Figure 4, we illustrate D_0, D_1, and D_2 for the case $m = 3$. Notice how D_{n+1} is constructed by erasing m subintervals from each interval in D_n. This can be thought of as a generalization of the construction of the standard Cantor set.

To proceed, we define

$$C(i_0, i_1, \ldots, i_n) = \{x : i_j^x = i_j, \quad 0 \le j \le n, \quad |i_{n+1}^x| > m\}$$

and

$$C_n = \bigcup_{\substack{i_0,i_1,\ldots,i_n \\ \forall j\le n, \quad |i_j|\le m}} C(i_0, i_1, \ldots, i_n).$$

That is, C_n is the set of points in D_n that are not in D_{n+1}.

With this definition we have $C_n \subseteq D_n$ and $D_{n+1} = D_n \setminus C_n$. We will show that there is a constant $u > 0$ such that

$$\frac{\mu(C_n)}{\mu(D_n)} \ge u. \tag{28}$$

This will prove our assertion since it implies that

$$\begin{aligned}\mu(D_n) &= \mu(D_{n-1}) - \mu(C_{n-1}) \le (1-u)\mu(D_{n-1}) \le \cdots \\ &\le (1-u)^n \mu(D_0),\end{aligned}$$

and $\mu(D_n) \downarrow 0$.

We will actually prove something stronger, that is, for *any* alternating sequence $i_0, i_1, \ldots, i_n$

$$\frac{\mu(C(i_0, i_1, \ldots, i_n))}{\mu(D(i_0, i_1, \ldots, i_n))} \ge u$$

and (28) will follow because the $D()$ partition D_n and the $C()$ partition C_n.

We again use our old trick of constructing

$$D(i_0, i_1, \ldots, i_n) = g^{(|i_0|-1)}(h(D(i_1, i_2, \ldots, i_n))). \tag{29}$$

Similarly,

$$C(i_0, i_1, \ldots, i_n) = g^{(|i_0|-1)}(h(C(i_1, i_2, \ldots, i_n))). \tag{30}$$

We have already seen that $D(i_0, i_1, \ldots, i_n)$ is an interval. Similarly, $C(i_0, i_1, \ldots, i_n)$ is an interval. This follows from (30) and the fact that $C(i_n)$ is an interval. We can therefore apply (19) twice to get

$$\frac{\mu(C(i_0,i_1,\ldots,i_n))}{\mu(D(i_0,i_1,\ldots,i_n))} = \frac{\mu(C(i_1,i_2,\ldots,i_n))}{\mu(D(i_1,i_2,\ldots,i_n))} \cdot \frac{v'(\zeta_1)}{v'(\zeta_2)} \qquad \zeta_1,\zeta_2 \in D(i_0,i_1,\ldots,i_n) \tag{31}$$

where $v(x) = g^{(|i_0|-1)}(h(x))$.

Let $r = 1 - (a_2^2+1)^{-1/2} < 1$. We will now show that there is some constant c such that $cr^n < 1$ for n large enough, that is,

$$\left|\frac{v'(\zeta_1)}{v'(\zeta_2)}\right| \le 1 - cr^n.$$

When $x \neq 0$ both $g(x)$ and $h(x)$ are doubly continuously differentiable functions with bounded first and second derivatives so $v(x)$ is as well. Recall that we are considering only points whose forward orbits lie in $[-a_m, a_m]$. We may therefore assume that

$$\frac{1}{a_{m+1}} \le |\zeta_1|, |\zeta_2| \le a_m.$$

Let

$$c_1 = \min_{1/a_{m+1} \le |x| \le a_m.} |v''(x)|.$$

Recall that Lemma 2 implies $|\zeta_1 - \zeta_2| \le 2^m r^n$. Together with Taylor's theorem with remainder this yields

$$|v'(\zeta_1) - v'(\zeta_2)| \le c_1|\zeta_1 - \zeta_2| \le c_1 2^m r^n. \tag{32}$$

Let

$$c_2 = \min_{1/a_{m+1} \le |x| \le a_m} |v'(x)|.$$

By the definition of v we have $c_2 > 0$. If n is large enough that $c_1 2^m r^n / c_2 < 1$, then (32) implies that

$$\left|\frac{v'(\zeta_1)}{v'(\zeta_2)}\right| \ge 1 - \left|\frac{v'(\zeta_1) - v'(\zeta_2)|}{v'(\zeta_2)}\right| \ge 1 - cr^n$$

where $c = c_1 2^m / c_2$. Now set

$$u_n = \min_{i_0,i_1,\ldots,i_n} \frac{\mu(C(i_0,i_1,\ldots,i_n))}{\mu(D(i_0,i_1,\ldots,i_n))}$$

and let N be such that $cr^N < 1$. Substituting into equation (31) tells us that $u_n > u_{n-1}(1 - cr^n)$. Telescoping this inequality yields

$$u_n \ge u_N \prod_{N<t\le n} (1 - cr^t).$$

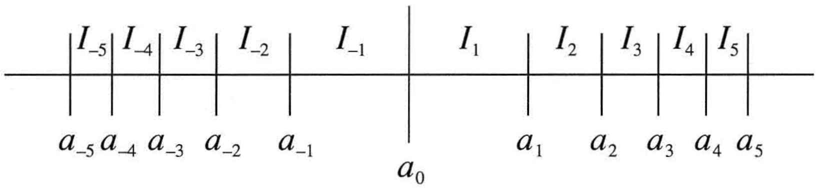

Figure 5: This figure illustrates how the intervals I_i partition $\Re$ when $f(x) = x - 1/x$.

The sum $\sum_t r^t$ converges and therefore the product $\prod_{N<t}(1 - cr^t)$ converges to some constant greater than 0. Furthermore, we know that for all n, $u_n > 0$. Therefore, $u = \inf u_n$ exists and is greater than 0. ∎

To review, in this section we have exhibited a one-to-one correspondence between sequences of alternating integers and points on the real axis. This correspondence, given by Theorem 1, was used to derive many properties of the iterated dynamics of f. Basically, the theorem showed that a point is uniquely defined by its dynamic behavior and there exists a point corresponding to every behavior.

4. Dynamics of $f(z) = z - 1/z$

In this section, we sketch how to modify the analysis of the previous section to analyze the iterates of

$$f(x) = x - \frac{1}{x}. \tag{33}$$

The analysis is almost the same as that of (1). The only difference is in the definition of the functions $g(x)$ and $h(x)$. The inverses of (33) are

$$g(x) = \begin{cases} \frac{x+\sqrt{x^2+4}}{2}, & x \geq 0 \\ \frac{x-\sqrt{x^2+4}}{2}, & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} \frac{x-\sqrt{x^2+4}}{2}, & x \geq 0 \\ \frac{x+\sqrt{x^2+4}}{2}, & x < 0 \end{cases}.$$

Otherwise the analysis is exactly the same (although some of the constants differ). Figure 5 shows the partition $\Re$ by the I_i under the new definitions. Notice that whereas in the previous section $\mu(I_i) \sim 2^{|i|}$, here $\mu(I_i) \sim \ln |i|$. This does not cause any changes in the analysis.

Acknowledgments

The first author was supported in part by National Science Foundation grant DCR-8605962 and Office of Naval Research grant N00014-87-K-0460. He was later supported by a Chateaubriand fellowship from the French Ministère des Affaires Étrangères and by the European Community, Esprit II Basic Research Action Number 3075 (ALCOM).

The second author was supported in part by National Science Foundation grant DMC-8451214.

The authors thank Ronald Graham for bringing the question of the dynamics of $f(z) = z - 1/z$ to their attention. They also thank an anonymous referee for detailed comments.

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