Newton’s Method for Quadratics, and Nested Intervals

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Abstract. The function $f_1(z) = \frac{1}{2}(z - 1/z)$ is the Newton map associated with the quadratic polynomial $z^2 + 1$. The iterated dynamics of $f$ on $\mathbb{R}$ is usually studied by exploiting the conjugacy between $f$ and the function $z \mapsto z^2$. In this paper we show how to use the technique of nested intervals to yield a direct analysis of the dynamics. We also show how to use the same technique to analyze the dynamics of the function $f_2(z) = z - 1/z$.

1. Introduction

In this note we will show how to utilize the technique of nested intervals to examine the iterated dynamics of two related functions that map $\mathbb{R}$, the set of real numbers, to itself. The two functions are

$$f_1(z) = \frac{1}{2}(z - 1/z) \quad (1)$$

and

$$f_2(z) = z - 1/z. \quad (2)$$

By studying the iterated dynamics of a function $f$ we mean studying the infinite sequences generated by iterating the function $f$ on initial seeds $x_0$, that is,

$$x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots, x_i = f(x_{i-1}), \ldots, \quad i = 1, 2, \ldots (3)$$

This sequence is sometimes known as the forward orbit of $x_0$. For $f = f_1$ this sequence has already been extensively studied because $f_1$ is the Newton map used to find the roots of the quadratic function $g(z) = z^2 + 1$. That is, given an initial seed $x_0$, the sequence (3) will usually converge to either $i$ or
-i, the two roots of $g$. The set containing all $x_0$ for which the sequence (3) does not converge is the Julia set associated with $f$. The Julia set associated with both $f_1$ and $f_2$ is the real line $\mathbb{R}$.

The usual method of analyzing the iterated dynamics of (1) uses the fact that $f_1$, considered as a mapping on the Riemann sphere, is conjugate under a linear fractional transformation to the function $z \mapsto z^2$. The purpose of this note is to illustrate how to analyze the dynamics directly without using the conjugacy relationship. The only tools we use are simple ones from elementary calculus and point-set topology. We exhibit a one-to-one correspondence between points in $\mathbb{R}$ and integer sequences of alternating sign, for example, $5, -7, 4, -6, \ldots$. This correspondence has some nice properties. For example, if the sequence associated with $x_0$ is periodic/bounded; then (3) is also periodic/bounded. Furthermore, if the associated sequences of a series of points converges to the sequence associated with $x$, then the points themselves converge to $x$. We will use these properties to analyze the dynamics of $f_1$. We will also show how this analysis, slightly modified, can be used to analyze the dynamics of $f_2$.

In section 2 we quickly review Newton’s method for finding the roots of polynomials. We also briefly sketch the conjugacy mapping that forms the basis for the usual analysis of $f_1$. In section 3 we explain what we mean by nested intervals. We then show how to use nested intervals to define the behavior of a point under $f_1$ in such a way that there is a one-to-one correspondence between points in $\mathbb{R}$ and possible behaviors. This correspondence will yield immediate proofs of the standard facts about the iterated behavior of $f$. As an example, it will show that the preimages of any point in $\mathbb{R}$ are dense in $\mathbb{R}$. It will also show that the set of points that have bounded forward orbits is ‘Cantor-like.’ Finally, in section 4 we will show how to modify the analysis of section 3 so that it can be applied to the iterated behavior of $f_2$.

2. Newton’s method

In this section we provide a quick review of Newton’s method for finding the roots of a polynomial. For a more complete explanation see [4]. Let $g(z) = \sum_{i=0}^{n} a_i z^i$ be an $n$th-degree polynomial. The Newton’s map associated with $g$ is

$$f(z) = z - \frac{g(z)}{g'(z)}.$$  \hspace{1cm} (4)

Newton’s method for finding a root of $g$ is to choose an initial seed $x_0$ and iterate $f$ on $x_0$ to construct the infinite sequence (3). Later we will need a more flexible notation: we set $f^{(0)}(x) = x$ and inductively define $f^{(i)}(x) = f(f^{(i-1)}(x))$ for $i \geq 1$. In this new notation (3) is written as

$$x_0, f(x_0), f^{(2)}(x_0), \ldots, f^{(i)}(x_0), \ldots.$$  \hspace{1cm} (5)
It is known that if \( x_0 \) is close enough to a root \( \alpha \) of \( g \), then the sequence \( x_i \) will converge to \( \alpha \). The set containing all points \( x_0 \in \mathbb{R} \) for which this sequence doesn't converge to some root of \( g \) is the Julia set associated with \( f \) [3].

As an example, suppose that \( g(z) = (z - \alpha)^2 \) where \( \alpha \in \mathbb{C} \) is an arbitrary complex number. Then

\[
f(z) = \frac{z + \alpha}{2}. \tag{6}\]

Therefore, for all \( z \in \mathbb{C} \),

\[
|\alpha - f(z)| = \frac{|\alpha - z|}{2}, \tag{7}\]

the sequence \( x_i \to \alpha \) irrespective of the original seed \( x_0 \), and the Julia set associated with \( f \) is empty.

If \( g \) is a quadratic with two distinct roots the situation is more interesting. Suppose, for example, that \( g(z) = z^2 + 1 = (z - i)(z + i) \). Then

\[
f(z) = \frac{1}{2} \left( z - \frac{1}{z} \right). \tag{8}\]

The classical method for analyzing \( f \) uses the fact that it is conjugate to the function \( h(z) = z^2 \). By conjugate we mean that there is some function \( T \) with a two-sided inverse \( T^{-1} \) such that

\[
f(z) = T^{-1} \circ h \circ T(z) \tag{9}\]

where \( \circ \) is the functional composition operator. By a two-sided inverse we mean that \( T(T^{-1}(z)) = T^{-1}(T(z)) = z \). It is straightforward to check that equation (9) is true when \( T(z) = (z + i)/(z - i) \). This \( T \) has the two-sided inverse \( T^{-1}(z) = i(z + 1)/(z - 1) \).

Iterating (9) yields

\[
x_i = T^{-1} \circ h^{(i)} \circ T(z), \tag{10}\]

which may be proved by induction. Therefore, the behavior of the sequence (5) can be examined by studying the behavior of the sequence \( z, z^2, z^4, z^8, \ldots, z^{2^i}, \ldots \).

First let \( S^1 = \{ z : |z| = 1 \} \) be the unit circle, \( \mathbb{H}^+ = \{ x + iy : x, y \in \mathbb{R}, y > 0 \} \) be the half-plane with positive imaginary part, and \( \mathbb{H}^- = \{ x - iy : x, y \in \mathbb{R}, y > 0 \} \) be the half-plane with negative imaginary part. The behavior of \( T \) is summarized in the following table.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( i )</th>
<th>(-i)</th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{H}^+ )</th>
<th>( \mathbb{H}^- )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(z) )</td>
<td>( \infty )</td>
<td>0</td>
<td>( 1 )</td>
<td>( S^1 )</td>
<td>( { z :</td>
</tr>
</tbody>
</table>

If \( |z| < 1 \), then \( z^{2^i} \) converges to 0. If \( |z| > 1 \) then \( z^{2^i} \) goes to infinity. If \( z \in S^1 \), the unit circle, then every point in \( z^{2^i} \) is also on \( S^1 \). If \( x_0 \in \mathbb{H}^+ \),
then $z = |T(x_0)| > 1$, $z^{2^i}$ tends to infinity, and $f^{(i)}(x_0) \rightarrow i$. Similarly, if $x_0 \in H^-$, then $z = |T(x_0)| < 1$, $z^{2^i}$ tends to 0, and $f^{(i)}(x_0) \rightarrow -i$. Finally, if $x_0 \in \mathbb{R}$, then $z = T(x_0) \in S^1$ and $z^{2^i}$ is contained in $S^1$. Therefore the behavior of the sequence $f^{(i)}(x_0)$ can be studied by examining the behavior of the sequence $z^{2^i}$ for $z \in S^1$.

In general, if $g(z) = (z - \alpha)(z - \beta)$ is a quadratic with two roots, then the analysis given above can be adapted to examine its associated Newton's function $f(z) = z - g(z)/g'(z)$ [4]. For more information on conjugacy methods and their applications to the study of iterated dynamics see [2].

In the next section we will show how to examine the dynamics of (5) without using the conjugacy relationship.

3. Nested intervals

A sequence of closed bounded intervals, $D_i = [a_i, b_i] \subseteq \mathbb{R}, i = 1, 2, 3, \ldots$ is said to be nested if $D_i \subseteq D_{i-1}$ for $i > 1$. An important fact about such sequences of intervals, one which often occurs in the analysis of dynamical systems [5], is that the intersection $\bigcap_i D_i$ is a nonempty closed interval, possibly a single point. In this section we show how to utilize the technique of nested intervals to analyze the dynamics of (1). That is, for $x \in \mathbb{R}$ we will study the behavior of (5).

Examination of $f$ shows that it is a monotonically increasing surjective function from $[1, \infty)$ onto $[0, \infty)$ and from $(0, 1]$ to $(-\infty, 0]$. Similarly, it is a monotonically increasing surjective function from $(-\infty, -1]$ onto $(-\infty, 0]$ and from $[-1, 0)$ onto $[0, \infty)$. Therefore, the infinite sequence $f^{(i)}(x)$ will forever alternate between positive decreasing and negative increasing sequences (Figure 1) unless, for some $i$, $f^{(i)}(x) = 0$. In this case we say that the sequence terminates since $f(0)$ is undefined. (Alternatively we can say that $f(0) = f(\infty) = \infty$ and $f^{(j)}(x) = \infty$ for $j > i$.)

The rest of this section will be given over to showing that there is a one-to-one correspondence between points in $\mathbb{R}$ and integer sequences of alternating sign

$$i_0, i_1, i_2, i_3, \ldots \quad \text{such that } i_j i_{j+1} \leq 0, \quad \text{for } j \geq 0$$
for example, 1, -3, 5, -7, . . . or -4, 6, -7, 5, . . . . This correspondence possesses properties useful in the analysis of the iterated sequence (5). For example, (5) is bounded if and only if the alternating integer sequence associated with $x$ is bounded.

We associate with each point $x \in \mathbb{R}$ two sets: $F_x$ and $B_x$. The set $F_x$, the forward orbit of $x$, is the set containing all of the iterates $f^{(i)}(x)$, while $B_x$, the backward orbit of $x$, is the set of all preimages of $x$. Formally

$$F_x = \{ f^{(i)}(x) : i = 0, 1, 2, \ldots \}$$
$$B_x = \{ z : \exists i, f^{(i)}(z) = x \}.$$ 

A technical note: If $z$ is a preimage of 0 ($z \in B_0$), then there is some $i$ such that $f^{(i)}(z) = 0$. In this case $f^{(i+1)}(z)$ is undefined and we define $F_z$ to be the finite set $\{ f^{(j)}(z) : 0 \leq j \leq i \}$.

Definition 1. A point $x \in \mathbb{R}$ is bounded if the forward orbit of $x$ is bounded, that is, if there is some constant $k$ such that $F_x \subset [-k, k]$.

Definition 2. A point $x \in \mathbb{R}$ is periodic if there is some integer $i \geq 1$ such that $f^{(i)}(x) = x$.

In this section we will prove the following facts:

Fact 1. $\forall x \in \mathbb{R}$, the set $B_x$ is dense in $\mathbb{R}$.

Fact 2. The set $\{ x : x$ is periodic$\}$ is dense in $\mathbb{R}$.

Fact 3. The set $\{ x : F_x$ is dense in $\mathbb{R}\}$ is itself dense in $\mathbb{R}$ and has cardinality $\aleph_1$.

Fact 4. The set $\{ x : x$ is bounded$\}$ is dense in $\mathbb{R}$, has cardinality $\aleph_1$ and measure 0. In fact, it is the union of a countable number of “Cantor-like” sets.

Notice that Fact 1 implies that the system that we are studying is chaotic. That is, it tells us that specifying a point $x$ to a very high degree of precision is not enough to tell us how $x$ behaves under iteration.

We define a set of intervals that partition the real line. First, for any given $x$ the equation $f(z) = x$ has exactly two solutions: $z_\pm = x \pm \sqrt{x^2 + 1}$. Furthermore $f(-1/z) = f(z)$, so $z_- = -1/z_+$. Thus one of $z_\pm$ has absolute value greater than or equal to 1 while the other has the opposite sign and absolute value less than or equal to 1. We will denote these two solutions by $g(x)$ and $h(x)$. These functions are defined in such a way that (for $x \neq 0$) $g(-x) = -g(x)$ and $h(-x) = -h(x)$:

$$g(x) = \begin{cases} x + \sqrt{x^2 + 1}, & x \geq 0 \\ x - \sqrt{x^2 + 1}, & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} x - \sqrt{x^2 + 1}, & x \geq 0 \\ x + \sqrt{x^2 + 1}, & x < 0 \end{cases}.$$ 

Notice that $g$ and $h$ are both monotonically increasing functions in the intervals $(-\infty, 0)$ and $(0, \infty)$. Thus, if $D$ is a bounded interval not containing
0, then both $g(D)$ and $h(D)$ will be bounded intervals that do not contain 0. These facts will be important later.

Next, we define a doubly infinite sequence $a_i$ as follows.

$$a_0 = 0, \quad a_i = g(a_{i-1}), \quad i > 0, \quad a_i = -a_{-i}, \quad i < 0.$$  

This permits us to define a set of intervals that partition $\mathbb{R}$. Set $I_0 = [0, 0]$ and

$$I_i = \begin{cases} (a_{i-1}, a_i), & i > 0 \\ (a_i, a_{i+1}), & i < 0 \end{cases}.$$  

Note that $I_i = -I_{-i}$ (see Figure 2). The functions $f$, $g$, and $h$ operate rather remarkably on the intervals $I_i$. Recall that $f$ is a monotonically increasing function from $[1, \infty)$ onto $[0, \infty)$ and $f(a_i) = a_{i-1}$ for $i \geq 1$. It follows that $f(I_i) = I_{i-1}$ for $i > 1$. (We use the notation $f(S) = \{f(x) : x \in S\}$ where $S$ is an arbitrary set.) Symmetrically, $f(I_i) = I_{i+1}$ for $i < 1$. There are similar results for $g$ and $h$. We summarize them in the following table:

$$
\begin{array}{c|c|c|c|c}
  i & 1 & -1 & > 1 & < -1 \\
  f(I_i) &=& (-\infty, 0) & (0, \infty) & I_{i-1} & I_{i+1} \\
  g(I_i) &=& I_2 & I_{i+1} & I_{i-1} \\
  h(I_i) \subseteq &= I_{-1} & I_1 & I_{-1} & I_1 \\
\end{array}
$$

We can associate with every $x \in \mathbb{R}$ a unique interval $I_i$ that contains it (because the $I_i$ form a partition of $\mathbb{R}$) and thus with the iterated sequence

$$x, f(x), f^2(x), \ldots, f^n(x), \ldots$$
we can associate a unique sequence of integers

\[ i_0, i_1, i_2, \ldots, i_j, \ldots \]

such that \( f^{(j)}(x) \in I_{i_j} \). As an example the point \( x = x_0 \) pictured in Figure 2 has the associated sequence

\[ 2, 1, -3, -2, -1, 3, 2, 1, -1, \ldots \]

(13)
because \( x \in I_2, f(x) \in I_1, f^{(3)}(x) \in I_{-3} \), and so forth. If \( x \in B_0 \), then its associated integer sequence is finite and ends at 0 since \( I_0 = \{0\} \) and \( f(0) \) is undefined. For example, if \( x = g(g(h(a_3))) \), then its associated sequence is

\[ -3, -2, -1, 5, 4, 3, 2, 1, 0. \]

(14)

If \( x \notin B_0 \), then the sequence associated with \( x \) is infinite.

Notice that both (13) and (14) share a peculiar property: they are composed of concatenations of positive subsequences that step down by one to 1 and negative subsequences that step up by one to -1. This property is a direct result of the first row of (12). More specifically, if \( x \in I_i, i > 1 \), then \( f(x) \in I_{i-1}, \ldots, f^{(i-1)}(x) \in I_1 \) and \( f^{(i)}(x) \in I_j \) where \( j \leq 0 \). Symmetrically, if \( x \in I_i, i < -1 \), then \( f(x) \in I_{i+1}, \ldots, f^{(|i|-1)}(x) \in I_{-1} \) and \( f^{(|i|)}(x) \in I_j \) where \( j \geq 0 \). The process terminates if and only if \( i = 0 \), since then \( x = 0 \) and \( f(0) \) is undefined.

What we have just described is a more detailed description of the "flip-flopping" behavior illustrated in Figure 1. In the next few pages (culminating in Theorem 1) we will show that there is a one-to-one correspondence between sequences of flip-flops and points in \( \mathbb{R} \). We will then use this correspondence to prove Facts 1, 2, 3, and 4.

Thus, given \( x \in I_i \), the first iterate of \( f \) on \( x \) whose location is unknown is \( f^{(|i|)}(x) \). The sequence associated with \( x \) can therefore be reconstructed from a sequence containing only the first elements of the increasing (decreasing) sequences, that is, \( 2, -3, 3, -1, \ldots \) in place of (13) and \(-3, 5, 0 \) in place of (14). We will call this abbreviated sequence the behavior of the sequence associated with \( x \), or simply the behavior of \( x \). We denote the behavior by \( S_x \). The formal definition of \( S_x \) follows.

**Definition 3.** For a point \( x \in \mathbb{R} \) its behavior, \( S_x \), is the sequence defined recursively as follows. Let \( i \) be the index of the interval containing \( x \), that is, \( x \in I_i \).

1. If \( i = 0 \), then \( S_x \) is the one item sequence 0.
2. If \( i \neq 0 \), then \( S_x = i, S_{f^{(|i|)}(x)} \).

For a given point \( x \) we will denote \( S_x \) by

\[ S_x = i_0, i_1, i_2, \ldots \]

From the definition we see that

\[ f^{(|i_0|+|i_1|+\cdots+|i_{n-1}|)}(x) \in I_{i_n}. \]
**Definition 4.** A sequence of nonzero integers $i_0, i_1, i_2, \ldots$ (finite or infinite) alternates if $\text{sign}(i_j) = -\text{sign}(i_{j+1})$, $j = 0, 1, 2 \ldots$.

**Definition 5.** An integer sequence $S$ that satisfies one of the following two conditions will be called a legal behavior.

1. $S$ is an infinite alternating sequence; or
2. $S = S', 0$ where $S'$ is a finite alternating sequence.

For example, $S = 1, -1, 1, 0$ and $1, -2, 3, -4, 5, -6, \ldots$ are legal behaviors, whereas $1, -1$ and $1, 1, 1, 1, 1, \ldots$ are not. We will use the notation $S = i_0, i_1, i_2, i_3, \ldots$ to represent the component integers of $S$. Thus the behavior $S = 1, -2, 3, -4, 5, -6, \ldots$ can be expressed by writing $i_k = (-1)^k(k + 1)$, $k \geq 0$.

**Definition 6.** Let $i_0, i_1, \ldots, i_n$ be a finite alternating sequence. Then

$$D(i_0, i_1, \ldots, i_n) = \{x : i_k^x = i_k, \quad 0 \leq k \leq n\}$$

is the set of all $x$ such that the first $n + 1$ components of $S_x$ are identical with $i_0, i_1, \ldots, i_n$.

For example, $D(2, -3, 3)$ contains all points $x \in I_2$ such that $f^{(2)}(x) \in I_{-3}$ and $f^{(2+3)} \in I_3$. This is illustrated in Figure 3. The set $D(i_0, i_1, \ldots, i_n)$ can be constructed explicitly as follows. First note that $D(i_0) = \{x : x \in I_{i_0}\} = I_{i_0}$. We also have the following lemma.

**Lemma 1.** Suppose $i_0, i_1, \ldots, i_n$ is an alternating sequence. Then

$$g(D(i_0, i_1, \ldots, i_n)) = \begin{cases} D(i_0 + 1, i_2, \ldots, i_n), & i_0 > 0 \\ D(i_0 - 1, i_2, \ldots, i_n), & i_0 < 0 \end{cases}$$

and

$$h(D(i_0, i_1, \ldots, i_n)) = \begin{cases} D(-1, i_0, i_2, \ldots, i_n), & i_0 > 0 \\ D(1, i_0, i_2, \ldots, i_n), & i_0 < 0. \end{cases}$$

An immediate application of Lemma 1 is that for $i_0, i_1, \ldots, i_n$, an alternating sequence

$$D(i_0, i_1, \ldots, i_n) = g^{(|i_0|-1)}(h(D(i_1, i_2, \ldots, i_n))).$$

As an example, we show how to apply the lemma to construct the intervals illustrated in Figure 3.

1. $D(2) = I_2$;
2. $D(-3) = I_{-3}$ so $D(2, -3) = g(h(I_{-3}))$;
3. $D(3) = I_3$ so $D(-3, 3) = g^{(2)}(h(I_3))$ and $D(2, -3, 3) = g(h(D(-3, 3))) = g(h(g^{(2)}(h(I_3))))$. 
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Figure 3: This figure exhibits how the nested intervals converge to a point. In the example we show the first three steps in the process of finding a point \(x\) with \(S_x = 2, -3, 3, \ldots\). The top row shows \(D(2)\), the middle row \(D(2, -3)\), and the bottom row \(D(2, -3, 3)\).

In general, to explicitly construct \(D(i_0, i_1, \ldots, i_n)\) we set
\[
t_0(x) = x, \quad t_n(x) = t_{n-1} \circ g^{(|i_{n-1}| - 1)} \circ h(x).
\] (18)

Repeated application of (17) yields
\[
D(i_0, i_1, \ldots, i_n) = t_n(D(i_n)) = t_n(I_{i_n}).
\]

As mentioned before, if \(D\) is an interval not containing zero, then so are \(h(D)\) and \(g(D)\); thus \(t_n(D)\) is an interval not containing zero. This proves that \(D(i_0, i_1, \ldots, i_n)\) is an interval that does not contain zero. Furthermore, \(h\) and \(g\) are one-to-one functions so \(t_n\) is a one-to-one function that maps \(I_n\) onto \(D(i_0, i_1, \ldots, i_n)\). This will be important in the proof of Theorem 1.

Another consequence of Lemma 1 is an upper bound on the size of \(D(i_0, i_1, \ldots, i_n)\) that decreases geometrically with \(n\). We use \(\mu(D)\) to denote the standard Lebesgue measure of set \(D\).

Lemma 2. Set \(r = 1 - (a^2 + 1)^{-1/2} \approx .6173\). For \(i_0, i_1, \ldots, i_n\), an alternating sequence,
\[
\mu(D(i_0, i_1, \ldots, i_n)) \leq 2^{\|i_0\|} \cdot r^n.
\]

Proof. Our main tool will be the following variant of the the mean value theorem: if \(D\) is an interval and \(s\) a continuously differentiable function, then
\[
\mu(s(D)) = |s'(\xi)| \cdot \mu(D) \text{ for some } \xi \in D.
\] (19)
As a first application, notice that \( h \circ g \circ f(x) = h(-1/x) \) for \( 0 < x < 1 \). Since, for \( x \neq 0 \), we have \( h'(x) = 1 - |x|/\sqrt{1 + x^2} \), we immediately find that for all \( 0 < x < 1 \)

\[
(h \circ g \circ f)'(x) = \frac{\sqrt{1 + x^2} - 1}{x^2 \sqrt{1 + x^2}} < \frac{\sqrt{1 + x^2} - 1}{x^2} \leq \frac{1}{2}.
\]

Applying the mean value theorem gives that for any interval \( D \subseteq (0, 1) \),

\[
\mu(h \circ g \circ f(D)) < \frac{1}{2} \mu(D). \quad (20)
\]

To begin, let \( i > 0 \). Then from Lemma 1 and the equation \( f(h(x)) = x \) we have

\[
D(-1, i) = h(I_i) \\
= h(g^{i-1}(I_1)) \\
= (h \circ g \circ f)^{(i-1)}(h(I_1)).
\]

An immediate consequence is that \( \mu(D(-1, i)) \leq 2^{1-i} \). Since \( D(-1, i) = -D(1, -i) \) we have just proven that for \( i_0, i_1 \) alternating, \( i_0 = \pm 1 \),

\[
\mu(D(i_0, i_1)) \leq 2^{1-|i_1|}. \quad (21)
\]

Next, let \( i_0, i_1, \ldots, i_n \) be an alternating sequence with \( i_0 = -1 \). Again using Lemma 1 we can write

\[
D(-1, i_1, \ldots, i_n) = h \circ g^{(i_1-1)}(D(1, i_2, \ldots, i_n)) \\
= (h \circ g \circ f)^{(i_1-1)}(h(D(1, i_2, \ldots, i_n))).
\]

Therefore,

\[
\mu(D(-1, i_1, \ldots, i_n)) \leq 2^{1-i_1} \mu(h((D(1, i_2, \ldots, i_n)))). \quad (22)
\]

But \( h'(x) = 1 - (|x|/\sqrt{1 + x^2}) \) and

\[
D(1, i_2, \ldots, i_n) \subseteq \left[ \frac{1}{a_{i_2 - 1}}, \frac{1}{a_{-i_2}} \right],
\]

so another application of the mean value theorem gives

\[
\mu(h(D(1, i_2, \ldots, i_n))) \leq \left(1 - \frac{1}{\sqrt{a_{i_2 + 1}^2 + 1}}\right) \mu(D(1, i_2, \ldots, i_n)). \quad (23)
\]

Substituting back into (22) and taking symmetry into account we have just proven that for \( i_0, i_1, \ldots, i_n \), an alternating sequence with \( i_0 = \pm 1 \),

\[
\mu(D(i_0, i_1, \ldots, i_n)) \leq \left(1 - (a_{|i_2| + 1})^{-1/2}\right) \mu\left(D\left(\frac{i_1}{|i_1|}, i_2, i_3, \ldots, i_n\right)\right). \quad (24)
\]
We unravel this inequality by recursively applying it to its own right-hand side \( n - 1 \) times and then apply (21) once, obtaining

\[
\mu(D(\pm 1, i_1, i_2, \ldots, i_n)) \leq \prod_{2 \leq j \leq n} 2^{1-|i_j|} \left( 1 - \frac{1}{\sqrt{a_{|i_j|+1}^2 + 1}} \right).
\]

We now examine this product on a term-by-term basis. If \( |i_j| = 1 \), then \( 1 - (a_{|i_j|+1}^2 + 1)^{-1/2} = r \), while if \( |i_j| > 1 \), then \( 2^{1-|i_j|} \leq 1/2 < r \). Therefore,

\[
\mu(D(\pm 1, i_1, i_2, \ldots, i_n)) \leq r^{n-1}.
\]

Until now we assumed that \( i_0 = \pm 1 \). We conclude the proof of the lemma by noting that, for arbitrary \( i_0 \),

\[
D(i_0, i_1, \ldots, i_n) = g^{(i_0-1)}(D(1, i_1, i_2, \ldots, i_n)).
\]

For all \( x \neq 0 \), we have \( g'(x) = 1 + |x|/\sqrt{1+x^2} \leq 2 \). The lemma therefore follows from another application of the mean value theorem. \( \blacksquare \)

**Definition 7.** Let \( S = i_0, i_1, i_2, \ldots \) be an infinite legal behavior. Let \( S^k = i_0^k, i_1^k, i_2^k, \ldots \) be a sequence of (finite or infinite) behaviors. We say that \( S^k \) converges to \( S \) (and write \( S^k \to S \)) if the \( S^k \) converge to \( S \) component-wise, that is,

\[
\forall j, \exists N_j \text{ such that } \forall k > N_j, \forall j' \leq j, \ i_{j'}^k = i_{j'}.
\]

For example, let \( S \) be the sequence \( S_j = (-1)^j (j + 1) \) and \( S^k \) the sequences \( S_j^k = (-1)^j ((j \mod k) + 1) \). Then

\[
S = 1, -2, 3, -4, 5, -6, \ldots
\]
\[
S^1 = 1, -1, 1, -1, 1, -1, 1, -1, \ldots
\]
\[
S^2 = 1, -2, 1, -2, 1, -2, 1, -2, \ldots
\]
\[
S^3 = 1, -2, 3, -1, 2, -3, 1, -2, \ldots
\]

and \( S_k \to S \).

We now formulate and prove our main theorem.

**Theorem 1.** The correspondence

\[
x \leftrightarrow S_x
\]

is a one-to-one and onto mapping between \( \mathbb{R} \) and the set of legal behaviors. Furthermore, if \( x \) is a point and \( x_i \) is a sequence of points such that \( S_{x_i} \to S_x \), then \( x_i \to x \).
Proof. To prove the first part, we must show that for every legal behavior $S$ there is a unique point $x$ such that $S_x = S$. We will treat the two cases $S$ finite and $S$ infinite separately.

First assume that $S$ is a finite alternating sequence

$$S = i_0, i_1, \ldots, i_n, 0.$$ 

Recall the function $t_n$ defined in (18), a one-to-one function with the property

$$D(i_0, i_1, \ldots, i_n) = t_n(I_{i_n}).$$

We claim that $x = t_n(a_{i_n})$ is the unique point such that $S_x = S$. This $x$ satisfies $S_x = S$ because $x \in D(i_0, i_1, \ldots, i_n)$ and $f^{|i_n|}(a_{i_n}) = 0$. Suppose now that $x \neq t_n(a_{i_n})$ is another point such that $S_x = S$. Then $x \in D(i_0, i_1, \ldots, i_n)$ so there is a unique $x' \in I_{i_n}$ such that $x = t(x')$. Furthermore, we must have $f^{|i_n|}(x') = 0$. The unique $x'$ that satisfies this last condition is $x' = a_{i_n}$. Therefore $x = t_n(a_{i_n})$ is the unique point satisfying $S_x = S$.

Now assume that $S = i_0, i_1, \ldots, i_n, \ldots$ is infinite. By definition,

$$S_x = S \iff x \in \bigcap_n D(i_0, i_1, \ldots, i_n).$$

To simplify our notation we set $D_n = D(i_0, i_1, \ldots, i_n)$. We have already seen that the $D_n$ are nested intervals with $\mu(D_n) \downarrow 0$. If the intervals were also closed, then as mentioned in the first paragraph in this section, there would be a unique $x$ such that $\cap_n D_n = \{x\}$ and we would be finished.

But the $D_n$ are not closed; they are half-open intervals such as $D(5) = [a_4, a_5]$. Thus $\overline{D_n}$, the closure of $D_n$, is closed, that is, $\overline{D(5)} = [a_4, a_5]$. Furthermore, $\mu(\overline{D_n}) = \mu(D_n) \downarrow 0$ so there is a unique $x$ such that $\cap_n \overline{D_n} = \{x\}$. To prove that there is a unique $x$ such that $S_x = S$, it therefore suffices to show that $\cap_n \overline{D_n} = \cap_n \overline{D_n}$.

Recall that $D(i_0, i_1, \ldots, i_n) = t_n(I_{i_n})$ where

$$t_0(x) = x, \quad t_n(x) = t_{n-1} \circ g^{|i_{n-1}|} \circ h(x).$$

The function $t_n$ is the repeated composition of the functions $h$ and $g$. Both of these functions are continuous except at $x = 0$ and map $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$. Thus

$$\overline{D(i_0, i_1, \ldots, i_n)} = t_n(I_{i_n})$$

$$= t_{n-1} \circ g^{|i_{n-1}|} \circ h(I_{i_n})$$

$$= t_{n-1} \circ g^{|i_{n-1}|} (h(I_{i_n})).$$

The next step is to calculate $h(I_i)$ for all $i \neq 0$. If $i > 0$, then $I_i = (a_{i-1}, a_i]$ and so $h(I_i) = \left( -\frac{1}{a_i}, -\frac{1}{a_{i+1}} \right]$. Thus $h(I_i) = h(I_i) \cup \{ -\frac{1}{a_i} \}$. A similar calculation shows that this remains true even when $i < 0$. Therefore we have shown that

$$\overline{D(i_0, i_1, \ldots, i_n)} = \overline{D(i_0, i_1, \ldots, i_n) \cup \{ t_{n-1} \circ g^{|i_{n-1}|} (u_n) \}}$$
where \( u_n = -1/a_{in} \). We can rewrite this as \( \overline{D_n} = D_n \cup \{x_n\} \) where \( x_n = t_{n-1} \circ g^{(|i_{n-1}|-1)}(u_n) \). By definition we have

\[
\begin{align*}
  f^{(|i_0|+|i_1|+\cdots+|i_{n-2}|+|i_{n-1}|-1)}(x_n) & = u_n \\
  & = -\frac{1}{a_{in}}.
\end{align*}
\]

and thus \( x_n \in B_0 \), the set of preimages of 0.

Suppose now that \( \{x\} = \cap_n \overline{D_n} \neq \cap_n D_n \). Because the \( D_n \) are nested there must be some integer \( N \) such that for all \( n \geq N \), \( x \in D_n \); but \( x \notin D_n \), so \( x = x_n \) for all \( n \geq N \). From what we have seen above we know that \( x = x_N \in B_0 \), so \( S_x \) is finite. Thus there is some \( m \) such that \( f^{(m)}(x) = 0 \); any further iterate of \( x \) will be undefined. In particular, \( f^{(|i_0|+|i_1|+\cdots+|i_{m-1}|+|i_m|-1)}(x) \) will be undefined. But this contradicts \( 25 \) so we must have \( \{x\} = \cap_n \overline{D_n} = \cap_n D_n \), and thus for every \( S \), there is a unique \( x \) with \( S_x = S \).

We now prove that \( S_{x_i} \to S_x \) implies \( x_i \to x \). We must show that for every \( \epsilon > 0 \) there is an \( N \) such that for all \( n \geq N \), \( |x_n - x| \leq \epsilon \). This is straightforward. Given \( s \) let \( j \) be the first integer such that \( \mu(D(i_0, i_1, \ldots, i_j)) \leq \epsilon \). Lemma 2 tells us that such a \( j \) must exist. Since \( S_{x_i} \to S_x \) there must be an \( N \) such that for all \( n \geq N \) and for all \( k \leq j \), \( i_{k+n} = i_k \). Therefore \( x_n, x \in D(i_0, i_1, \ldots, i_j) \) and so \( |x_n - x| \leq \epsilon \).

The theorem lets us derive properties describing the iterated dynamics of \( f \). We use the fact that if \( S_y \) is a suffix of \( S_x \), then \( y \in F_x \) and \( x \in B_y \). By suffix we mean that there is some \( n > 0 \) such that \( i^y_{j+n} = i^x_j \) for all \( j \geq 0 \). Theorem 1 tells us that for a given \( x \) and fixed \( n \) there is a unique \( y \) that fulfills this condition. The definition of \( S_x \) tells us that \( y = f^{(|i_0|+|i_1|+\cdots+|i_{n-1}|-1)}(x) \). Thus \( y \in F_x \) and \( x \in B_y \).

As an example, suppose that

\[
\begin{align*}
  S_y &= 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, -12, \ldots \\
  S_x &= 13, -13, 13, -13, 1, -2, 3, -4, 5, -6, 7, -8, \ldots 
\end{align*}
\]

Then \( S_y \) is a suffix of \( S_x \).

Recall that a point is periodic if there is some \( j \) such that \( f^{(j)}(x) = x \). The discussion in the previous paragraph implies that if \( S_x \) is periodic in the sense that there is some \( n > 0 \) such that \( i^x_{j+n} = i^x_j \) for all \( j \geq 0 \), then \( x \) is periodic (the converse is almost but not quite true). Thus we can prove Fact 2.

**Lemma 3.** Let \( P = \{x : \exists j > 0 \text{ such that } f^{(j)}(x) = x\} \) be the set of periodic points. Then \( P \) is dense in \( \mathbb{R} \).

**Proof.** We will actually show that \( P \) is dense in the set of all points with infinite behaviors, \( \mathbb{R} \setminus B_0 \). Since \( B_0 \) is countable the proof will follow. The general idea is to construct a sequence of periodic behaviors \( S_{x_n} \) that converge to \( S_x \). For example, if

\[
S_x = 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots
\]
then we might choose \( x_n \) so that
\[
S_{x_1} = 1, -2, 1, -2, 1, -2, 1, -2, 1, -2, \ldots \\
S_{x_2} = 1, -2, 3, -4, 1, -2, 3, -4, 1, -2, \ldots \\
S_{x_3} = 1, -2, 3, -4, 5, -6, 1, -2, 3, -4, \ldots \\
\]
and so forth.

Formally let \( x \in \mathbb{R} \setminus B_0 \) with \( S_x = i_0^x, i_1^x, \ldots \). Then by choice, \( S_x \) is infinite. For \( n > 0 \), let \( x_n \) be the unique point that satisfies \( i_{jn}^{x_n} = i_{j \text{mod}(2n)}^x \). Then
\[
S_{x_n} = i_0^{x_n}, i_1^{x_n}, \ldots, i_{2n-1}^{x_n}, i_0^{x_n}, i_1^{x_n}, \ldots, i_{2n-1}^{x_n}, \ldots.
\]
(The modulus is taken \( 2n \) and not \( n \) to ensure that \( S_{x_n} \) alternates.) By definition, \( x_n \in P \). It is not hard to see that \( S_{x_n} \rightarrow S_x \), so \( x_n \rightarrow x \) and we have finished the proof.

We use the same technique to prove Fact 1.

**Lemma 4.** For all \( x \in \mathbb{R} \), the set \( B_x \) is dense in \( \mathbb{R} \).

**Proof.** Fix \( x \in \mathbb{R} \). The behavior \( S_x = i_0^x, i_1^x, \ldots \) can be finite or infinite. As in the previous lemma it will be enough to show that \( B_x \) is dense in \( \mathbb{R} \setminus B_0 \).

Let \( y \) be an arbitrary point in \( \mathbb{R} \setminus B_0 \), that is, \( S_y = i_0^y, i_1^y, \ldots \) is infinite. We construct a sequence of points \( x_n \) such that \( S_{x_n} \rightarrow S_y \). Furthermore, \( S_x \) will be a suffix of each of the \( S_{x_n} \) so \( x_n \in B_x \). The proof of the lemma will follow from Theorem 1. As an example suppose that
\[
S_x = 1, -1, 1, -1, 1, -1, 1, -1, \ldots \\
S_y = 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots
\]
We can choose the \( x_n \) such that
\[
S_{x_1} = 1, -2, 1, -1, 1, -1, 1, -1, 1, -1, \ldots \\
S_{x_2} = 1, -2, 3, -4, 1, -1, 1, -1, 1, -1, \ldots \\
S_{x_3} = 1, -2, 3, -4, 5, -6, 1, -1, 1, -1, \ldots \\
S_{x_4} = 1, -2, 1, -1, 5, -6, 7, -8, 1, -1, \ldots
\]
Formally, we construct the \( x_n \) so that \( S_{x_n} \) starts out as \( S_y \) but ends as \( S_x \). To do this we define a parameter \( \delta \) that ensures that the copies of \( S_x \) commence at locations in the \( S_{x_n} \) that have the proper parity. Let
\[
\delta = \begin{cases} 
0 & \text{if } \text{sign}(i_0^x) = \text{sign}(i_0^y) \\
1 & \text{if } \text{sign}(i_0^x) \neq \text{sign}(i_0^y)
\end{cases}
\]
where \( \text{sign}(i) = i/|i| \). We now set \( x_n \) to be the unique point such that
\[
i_k^{x_n} = \begin{cases} 
i_k^y & 0 \leq k \leq 2n + \delta \\
i_k^{x_n} & k \geq 2n + \delta + 1
\end{cases}
\]
The points \( x_n \) are all in \( B_x \) and \( S_{x_n} \rightarrow S_y \), therefore \( x_n \rightarrow y \) and we have finished the proof.

Utilizing the same technique, we now prove Fact 3.
Lemma 5. The set 

\[ D = \{ x : F_x \text{ is dense in } \mathbb{R} \} \]

is itself dense in \( \mathbb{R} \) and has cardinality \( \aleph_1 \).

Proof. We start by showing how to construct a point \( x \in D \). Let \( A \) be the set of all finite alternating sequences with odd length whose first element is positive. It includes sequences such as \( \alpha = 1, -3, 5 \). For any sequence \( \alpha \), we write \(-\alpha\) to denote the sequence whose elements are the negatives of those in \( \alpha \). For the given example \(-\alpha = -1, 3, -5\). Now \( A \) is a countable set so we can enumerate all the sequences in \( A \) as \( \alpha_1, \alpha_2, \ldots \). Let \( S \) be the concatenation of all of the pairs \( \alpha_i, -\alpha_i \). That is,

\[ S = \alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_3, -\alpha_3, \ldots \]

Let \( x \) be the unique point with \( S \cdot x = S \). We claim that \( F_x \) is dense in \( \mathbb{R} \).

Let \( y \in \mathbb{R} \setminus B_0 \) and \( \epsilon > 0 \). We must show that there is some \( i \) such that \( |f^{(i)}(x) - y| < \epsilon \). From Lemma 2 we know that there is some \( n \) such that if \( y' \in D(\alpha^0_n, \alpha^1_n, \ldots, \alpha^n_n) \), then \( |y' - y| < \epsilon \). Now, by definition, if the sequence \( \alpha^0_n, \alpha^1_n, \ldots, \alpha^n_n \) appears anywhere in \( S \cdot x \), then there is some \( i \) such that \( f^{(i)}(x) \in D(\alpha^0_n, \alpha^1_n, \ldots, \alpha^n_n) \). By construction we know that every finite alternating sequence appears somewhere in \( S \cdot x \). Thus there is some \( i \) such that \( |f^{(i)}(x) - y| < \epsilon \). Since \( y \) and \( \epsilon \) were chosen arbitrarily we have just shown that \( F_x \) is dense in \( \mathbb{R} \).

It is easy to modify the construction to show that \( D \) has cardinality \( \aleph_1 \). For each \( i = 1, 2, 3, \ldots \) choose \( \beta_i \) to be one of the two sequences \( 1, -1 \) or \( 2, -2 \). Let \( S \) be the concatenation of all of the triplets \( \alpha_i, -\alpha_i, \beta_i \). That is,

\[ S = \alpha_1, -\alpha_1, \beta_1, \alpha_2, -\alpha_2, \beta_2, \alpha_3, -\alpha_3, \beta_3, \ldots \]

Let \( x \) be the unique point with \( S \cdot x = S \). The analysis of the previous paragraph shows that \( F_x \) is dense in \( \mathbb{R} \). Since there are \( \aleph_1 \) possible choices of the sequences \( \beta_1, \beta_2, \beta_3, \ldots \), there are at least \( \aleph_1 \) points \( x \) with \( F_x \in D \).

It remains to be shown that \( D \) itself is dense in \( \mathbb{R} \). This is trivial. If \( x \in D \), then \( f(x) \in D \) so \( F_x \subseteq D \). Since \( F_x \) is dense in \( \mathbb{R} \) so is \( D \). \( \blacksquare \)

We conclude this section by analyzing the structure of the set of all bounded points. This set will be shown to be the union of a countable number of sets, each possessing a structure similar to that of the Cantor set (Fact 4).

Theorem 2. The set of bounded points

\[ S = \{ x : \exists c > 0 \text{ such that } F_x \subseteq [-c, c] \} \]

has cardinality \( \aleph_1 \) and measure 0.
Figure 4: This figure illustrates the first three steps in the construction of $S_3$, the set of all points whose forward iterates are in the bounded interval $[-a_3, a_3]$. The unshaded area in the top diagram is $D_0$, in the middle $D_1$, and in the bottom $D_2$. At each step $D_{n+1} = D_n \setminus C_n$.

**Proof.** We define $S_m$, the set of points whose forward orbit is in $[-a_m, a_m]$:

$$S_m = \{ x : F_x \subseteq [-a_m, a_m] \}.$$  \hfill (26)

Since $a_m \uparrow \infty$ we have $S_1 \subseteq S_2 \subseteq S_3 \ldots$ and $S = \bigcup S_n$. We will prove that the cardinality of $S_2$ is $\aleph_1$ and therefore so is the cardinality of $S$. We will also prove that for every $m$, $\mu(S_m) = 0$ and thus $\mu(S) = \mu(\bigcup_m S_m) = 0$ because the countable union of sets of measure 0 has measure 0.

Our main tool will again be the correspondence $x \leftrightarrow S_x$. As before, for $x \in \mathbb{R}$ we denote

$$S_x = i_0, i_1, i_2, \ldots.$$  

With this notation it is easy to see that (26) can be rewritten as

$$S_m = \{ x : \forall j, |i_j^m| \leq m \}. \hfill (27)$$

For example, $S_1 = \{ x, -x \}$ where $x$ is the unique point such that

$$S_x = 1, -1, 1, -1, 1, -1, \ldots$$

In fact, $x = 1/\sqrt{3}$. (This point $x$ can be found by solving $f(x) = -x$.)

That $S_2$ (and therefore $S$) has cardinality $\aleph_1$ follows from Theorem 1 together with the fact that the set of infinite alternating sequences that can be constructed utilizing the integers $1, -1, 2, -2$ has cardinality $\aleph_1$. The second part, that $\mu(S_m) = 0$, will be more difficult to prove.
For the rest of the proof we assume that $m > 1$ is fixed. We set

$$D_n = \bigcup_{i_0, i_1, \ldots, i_n} D(i_0, i_1, \ldots, i_n)$$

where the union is taken over all alternating sequences of length $n + 1$. With this definition $S_m = \cap_n D_n$, so it will be enough to show that $\mu(D_n) \to 0$.

In Figure 4, we illustrate $D_0$, $D_1$, and $D_2$ for the case $m = 3$. Notice how $D_{n+1}$ is constructed by erasing $m$ subintervals from each interval in $D_n$. This can be thought of as a generalization of the construction of the standard Cantor set.

To proceed, we define

$$C(i_0, i_1, \ldots, i_n) = \{ x : i_j^x = i_j, \quad 0 \leq j \leq n, \quad |i_{n+1}^x| > m \}$$

and

$$C_n = \bigcup_{i_0, i_1, \ldots, i_n} C(i_0, i_1, \ldots, i_n).$$

That is, $C_n$ is the set of points in $D_n$ that are not in $D_{n+1}$.

With this definition we have $C_n \subseteq D_n$ and $D_n = D_n \setminus C_n$. We will show that there is a constant $u > 0$ such that

$$\frac{\mu(C_n)}{\mu(D_n)} \geq u. \quad (28)$$

This will prove our assertion since it implies that

$$\mu(D_n) = \mu(D_{n-1}) - \mu(C_{n-1}) \leq (1 - u)\mu(D_{n-1}) \leq \cdots \leq (1 - u)^n \mu(D_0),$$

and $\mu(D_n) \downarrow 0$.

We will actually prove something stronger, that is, for any alternating sequence $i_0, i_1, \ldots, i_n$

$$\frac{\mu(C(i_0, i_1, \ldots, i_n))}{\mu(D(i_0, i_1, \ldots, i_n))} \geq u$$

and (28) will follow because the $D()$ partition $D_n$ and the $C()$ partition $C_n$.

We again use our old trick of constructing

$$D(i_0, i_1, \ldots, i_n) = g(|i_0| - 1)(h(D(i_1, i_2, \ldots, i_n))). \quad (29)$$

Similarly,

$$C(i_0, i_1, \ldots, i_n) = g(|i_0| - 1)(h(C(i_1, i_2, \ldots, i_n))). \quad (30)$$
We have already seen that \( D(i_0, i_1, \ldots, i_n) \) is an interval. Similarly, \( C(i_0, i_1, \ldots, i_n) \) is an interval. This follows from (30) and the fact that \( C(i_n) \) is an interval. We can therefore apply (19) twice to get

\[
\frac{\mu(C(i_0, i_1, \ldots, i_n))}{\mu(D(i_0, i_1, \ldots, i_n))} = \frac{\mu(C(i_1, i_2, \ldots, i_n))}{\mu(D(i_1, i_2, \ldots, i_n))} \cdot \frac{v'(\xi_1)}{v'(\xi_2)} \quad (31)
\]

where \( v(x) = g^{(i_0|-1)}(h(x)) \).

Let \( r = 1 - (a_2^2 + 1)^{-1/2} < 1 \). We will now show that there is some constant \( c \) such that \( cr^n \) for \( n \) large enough, that is,

\[
\left| \frac{v'(\xi_1)}{v'(\xi_2)} \right| \leq 1 - cr^n.
\]

When \( x \neq 0 \) both \( g(x) \) and \( h(x) \) are doubly continuously differentiable functions with bounded first and second derivatives so \( v(x) \) is as well. Recall that we are considering only points whose forward orbits lie in \([-a_m, a_m]\). We may therefore assume that

\[
\frac{1}{a_{m+1}} \leq |\xi_1|, |\xi_2| \leq a_m.
\]

Let

\[
c_1 = \min_{1/a_{m+1} \leq |x| \leq a_m} |v''(x)|.
\]

Recall that Lemma 2 implies \( |\xi_1 - \xi_2| \leq 2^m r^n \). Together with Taylor's theorem with remainder this yields

\[
|v'(\xi_1) - v'(\xi_2)| \leq c_1 |\xi_1 - \xi_2| \leq c_1 2^m r^n. \quad (32)
\]

Let

\[
c_2 = \min_{1/a_{m+1} \leq |x| \leq a_m} |v'(x)|.
\]

By the definition of \( v \) we have \( c_2 > 0 \). If \( n \) is large enough that \( c_1 2^m r^n / c_2 < 1 \), then (32) implies that

\[
\left| \frac{v'(\xi_1)}{v'(\xi_2)} \right| \geq 1 - \left| \frac{v'(\xi_1) - v'(\xi_2)}{v'(\xi_2)} \right| \geq 1 - cr^n
\]

where \( c = c_1 2^m / c_2 \). Now set

\[
u_n = \min_{i_0, i_1, \ldots, i_n} \frac{\mu(C(i_0, i_1, \ldots, i_n))}{\mu(D(i_0, i_1, \ldots, i_n))}
\]

and let \( N \) be such that \( cr^N < 1 \). Substituting into equation (31) tells us that \( u_n > u_{n-1}(1 - cr^n) \). Telescoping this inequality yields

\[
u_n \geq u_N \prod_{N < t \leq n} (1 - cr^t).
\]
Figure 5: This figure illustrates how the intervals $I_i$ partition $\mathbb{R}$ when $f(x) = x - 1/x$.

The sum $\sum_i r^t$ converges and therefore the product $\prod_{N<\infty}(1 - cr^t)$ converges to some constant greater than 0. Furthermore, we know that for all $n$, $u_n > 0$. Therefore, $u = \inf u_n$ exists and is greater than 0.

To review, in this section we have exhibited a one-to-one correspondence between sequences of alternating integers and points on the real axis. This correspondence, given by Theorem 1, was used to derive many properties of the iterated dynamics of $f$. Basically, the theorem showed that a point is uniquely defined by its dynamic behavior and there exists a point corresponding to every behavior.

4. Dynamics of $f(z) = z - 1/z$

In this section, we sketch how to modify the analysis of the previous section to analyze the iterates of

$$f(x) = x - \frac{1}{x}.$$ \hspace{1cm} (33)

The analysis is almost the same as that of (1). The only difference is in the definition of the functions $g(x)$ and $h(x)$. The inverses of (33) are

$$g(x) = \begin{cases} \frac{x+\sqrt{x^2+4}}{2}, & x \geq 0 \\ \frac{x-\sqrt{x^2+4}}{2}, & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} \frac{x-\sqrt{x^2+4}}{2}, & x \geq 0 \\ \frac{x+\sqrt{x^2+4}}{2}, & x < 0 \end{cases}.$$\hspace{1cm} (33)

Otherwise the analysis is exactly the same (although some of the constants differ). Figure 5 shows the partition $\mathbb{R}$ by the $I_i$ under the new definitions. Notice that whereas in the previous section $\mu(I_i) \sim 2^{\lvert i\rvert}$, here $\mu(I_i) \sim \ln \lvert i\rvert$. This does not cause any changes in the analysis.

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References


