Mathematical Properties of Thermodynamic Automata

Guido Cosenza
Dipartimento di Scienze Fisiche, Università di Napoli,
Mostra d'Oltremare, Padiglione 19, I-80125 Napoli, Italy

Anthony Massobrio*
Luigi Sertorio†
Dipartimento di Fisica Teorica dell'Università di Torino, INFN,
Via P. Giuria 1, I-10125 Torino, Italy

Abstract. This paper concerns the study of some mathematical properties of Thermodynamic Automata. In the present paper, the governing linearized differential equations—for which a rigorous treatment is possible—are studied. The driving terms are assumed periodic and the controls are assumed piecewise constant. The explicit solution is found in the most general case with an arbitrary number of compartments composing the Thermodynamic Automaton and it is shown to be periodic with the same period of the driving terms. This property allows the derivation of interesting relationships among the time averages of the temperatures. Moreover, knowledge of the solutions enables proof that the trilinear system presented here and a bilinear system of equations introduced in a preceding paper are equivalent.

1. Introduction

We introduced in a preceding paper [1] the theory and the numerical simulation referring to the simplest Thermodynamic Automaton, called $Z(4,3)$. We denote with $Z(n,r)$ a compartmental model system described by n state variables (temperatures) and r bilinear controls acting on the heat flows across the interfaces between couples of compartments.

The content of this paper is the following.

In section 2 we model a boundary control according to which a given surface is either present or absent. As a consequence we have an interesting

---

*To whom correspondence should be sent. Electronic mail address: massobrio@torino.infn.it

†Present address: Service de Chimie Physique, ULB, Bruxelles, Belgium.

variable structure system with variable topology and corresponding variable number of the set of differential equations.

In section 3 the bilinear model equations introduced in reference [1] are linearized. If the system is linearized, an analytical study becomes possible.

In section 4 we study the mathematical properties of the linearized model in the general case of n compartments. We write the solution in a general form embodying the constant configurations that the control parameters can assume on (unspecified) finite time intervals. We show that if the driving terms are periodic, then the asymptotic behavior of a solution belonging to any initial condition is periodic with the same period. Within the same generality, we also show that the trilinear system can be reduced to a bilinear system.

In section 5 we return to the actual four-compartment system. We show that the continuous ensemble of controlled dynamical systems represented by $Z(4, 3)$ contains as a particular case a finite number of uncontrolled dynamical systems, in which the control parameters are constant on an infinite time interval. We integrate numerically the differential equations and obtain the average values in the eight constant control configurations; then we derive formal relationships among the average temperatures.

In section 6 we show numerically the accuracy of the linearization of the bilinear model and the equivalence between the bilinear and trilinear systems.

2. The variable structure system

We recall from reference [1] the problem of how to simulate the equilibration of temperature T_3 with the ambient thermal bath temperature T_E when the surface S_{3E} is removed.

The bilinear approach is discussed in reference [1].

In the trilinear model approach, c_{3E} is a control on the presence of the boundary S_{3E} between compartment 3 and the environment E: $c_{3E} = 0$ implies S_{3E} absent, $c_{3E} = 1$ implies S_{3E} present. The governing system of trilinear differential equations is then:

$$
\begin{aligned}
\mu_1 \dot{T}_1(t) &= [\beta_{12} + \tilde{\beta}_{12} \, c_{12}(t)] \, [T_2(t) - T_1(t)] \\
&\quad + \beta_{1E} \, [T_E(t) - T_1(t)] \\
\mu_2 \dot{T}_2(t) &= [\beta_{12} + \tilde{\beta}_{12} \, c_{12}(t)] \, [T_1(t) - T_2(t)] \\
&\quad + [1 - c_{3E}(t)] \, [\beta_{23} + \tilde{\beta}_{23} \, c_{23}(t)] \, [T_3(t) - T_2(t)] \\
&\quad + c_{3E}(t) \, [\beta_{23} + \tilde{\beta}_{23} \, c_{23}(t)] \, [T_E(t) - T_2(t)] \\
[1 - c_{3E}(t)] \, \mu_3 \dot{T}_3(t) &= [1 - c_{3E}(t)] \Big\{ [\beta_{23} + \tilde{\beta}_{23} \, c_{23}(t)] \, [T_2(t) - T_3(t)] \\
&\quad + \beta_{34} \, [T_4(t) - T_3(t)] + \beta_{3E} \, [T_E(t) - T_3(t)] \Big\} \\
\mu_4 \dot{T}_4(t) &= S_{4E}[\phi^{\text{in}}(t) - \varepsilon \, \sigma \, T_4^4(t)] \\
&\quad + [1 - c_{3E}(t)] \, \beta_{34} \, [T_3(t) - T_4(t)] \\
&\quad + c_{3E}(t) \, \beta_{34} \, [T_E(t) - T_4(t)] .
\end{aligned}
\tag{2.1}
$$

The trilinear terms are of the form $c_{3E}\, c_{jk}\, T_k$ or $c_{3E}\, c_{jk}\, T_j$.

When $c_{3E} = 1$ the differential equation of T_3 disappears and T_3 is substituted by the driving term T_E. This gives a system with a variable topology (compartment 3 merges with the external ambient E) and a state vector with a variable number of components ($n = 3$ when $c_{3E} = 1$, $n = 4$ when $c_{3E} = 0$). From the standpoint of mathematical physics, the bilinear model corresponds to a Newton flux boundary condition and the trilinear model to a Dirichlet boundary condition on S_{3E}.

The trilinear system (2.1) is intrinsically discontinuous. This fact raises analytical difficulties. However, in [1] we stated without proof that, in the limit $\tilde{\alpha}_{3E} \to \infty$, the bilinear system of equations is equivalent to the trilinear system. Furthermore it is immediate to verify that when $c_{3E} = 0$ the bilinear system is the same as system (2.1) without any passage to the limit. In section 4 we show in the general n-compartment case that the bilinear system contains the trilinear system as a particular case.

3. The Linearized System

We refer to the bilinear system of equations introduced in [1], section 3. The nonlinearity $T_4^4(t)$ in the fourth equation of [1], equation (3.3) (see also the fourth equation of the trilinear system (2.1)) is harmless. First of all the nonlinear term T_4^4 does not couple different equations; moreover $T_4(t)$ spans a bounded range of values due to the fact that the driving terms $T_E(t)$ and $g(t)$ are bounded. The linearization

$$T_4^4(t) \simeq 4\, T_0^3\, T_4(t) \; - 3\, T_0^4 \,, \tag{3.1}$$

where T_0 is a properly chosen temperature in the above-mentioned bounded range, gives the linear equations

$$\mu_i\, \dot{T}_i(t) = \sum_{j=1}^{4} a_{ij}(t)\, T_j(t) + g_i(t) \qquad (i = 1, \ldots, 4) \,, \tag{3.2}$$

where the driving terms are (see [1], equation (3.3) and appendix A)

$$\begin{cases} g_1(t) = \beta_{1E}\, T_E(t), \; g_2(t) = 0, \; g_3(t) = [\beta_{3E} + c_{3E}(t)\, \tilde{\beta}_{3E}]\, T_E(t), \\ g_4(t) = S_{4E}\left[g(t) + \sigma\, T_E^4(t) + 3\,\varepsilon\,\sigma\, T_0^4 \right] \end{cases} \tag{3.3}$$

and the matrix elements a_{ij} are

$$\begin{cases} a_{11} = -\beta_{12} - c_{12}\, \tilde{\beta}_{12} - \beta_{1E}, \; a_{12} = \beta_{12} + c_{12}\, \tilde{\beta}_{12}, \; a_{13} = a_{14} = 0; \\ a_{22} = -\beta_{12} - c_{12}\, \tilde{\beta}_{12} - \beta_{23} - c_{23}\, \tilde{\beta}_{23}, \; a_{23} = \beta_{23} + c_{23}\, \tilde{\beta}_{23}, \; a_{24} = 0; \\ a_{33} = -\beta_{23} - c_{23}\, \tilde{\beta}_{23} - \beta_{34} - \beta_{3E} - c_{3E}\, \tilde{\beta}_{3E}, \; a_{34} = \beta_{34}; \\ a_{44} = -4\, S_{4E}\, \varepsilon\, \sigma\, T_0^3 - \beta_{34}\,; \\ a_{ij} = a_{ji}. \end{cases} \tag{3.4}$$

4.   Mathematical properties of the linearized system

We first discuss the properties of the matrix $\mathbf{A}$ whose elements are listed in equation (3.4). $\mathbf{A}$ is symmetric and can be shown to be a stability matrix. We recall that a symmetric matrix is a stability matrix iff the determinants of the leading principal minors of have alternating signs; the first element a_{11} must be negative.

We expect system (3.2) to be stable because we are dealing with a compartmental reduction of the heat diffusion equation, which has stable solutions. After some calculations the reader may verify with the aid of equation (3.4) that the above-mentioned conditions are satisfied. Notice that $\mathbf{A}$ is always symmetric due to the scheme of the reduction.

Let us now consider the general case of n compartments with an $n \times n$ matrix $\mathbf{A}$ having negative eigenvalues λ_i. The dynamical equations are

$$\mu_i \dot{T}_i(t) = \sum_{j=1}^{n} a_{ij} T_j(t) + g_i(t) \qquad (i = 1, \ldots, n), \tag{4.1}$$

where the $g_i(t)$ are the driving terms.

To the purpose of having a symmetric matrix $\mathbf{A}$ we will work with the following set of equations:

$$\dot{T}_i^0(t) = \sum_{j=1}^{n} a_{ij} T_j^0(t) + g_i(\mu_i t) \qquad (i = 1, \ldots, n), \tag{4.2}$$

or in matrix form:

$$\dot{\mathbf{T}}^0 = \mathbf{A}\, \mathbf{T}^0(t) + \mathbf{g}(t), \tag{4.3}$$

where $\mathbf{g}$ is the vector having components $g_i(\mu_i t)$. It is easy to verify that

$$T_i(\mu_i t) = T_i^0(t) \qquad (i = 1, \ldots, n). \tag{4.4}$$

If $\mathbf{\Lambda}$ is the modal matrix of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ the resulting diagonal matrix,

$$\mathbf{A} = \mathbf{\Lambda}^{-1} \tilde{\mathbf{A}}\, \mathbf{\Lambda}, \tag{4.5}$$

we define

$$\mathbf{H}(t) \equiv \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & e^{\lambda_n t} \end{pmatrix} \tag{4.6}$$

and the state transition matrix at time t,

$$\mathbf{\Phi}(t) \equiv \mathbf{\Lambda}^{-1} \mathbf{H}(t)\, \mathbf{\Lambda}. \tag{4.7}$$

The solution of the associated scaled equation (4.3) is notoriously

$$\mathbf{T}^0(t) = \mathbf{\Phi}(t - t_0)\, \mathbf{T}^0(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - t')\, \mathbf{g}(t')\, dt', \tag{4.8}$$

which can also be written as

$$\mathbf{T}^0(t_0 + h) = \boldsymbol{\Phi}(h)\,\mathbf{T}^0(t_0) + \int\limits_0^h \boldsymbol{\Phi}(h-s)\,\mathbf{g}(s+t_0)\,ds\,. \tag{4.9}$$

More generally we may write the solution in the case in which the controls are constant in the time interval $[t_0, h_i)$, $\forall h_i > 0$ $(i = 1, \dots, n-1)$. Let $\boldsymbol{\Phi}^{(i)}$ be the transition matrix in $[t_0, h_i)$; then

$$
\begin{aligned}
\mathbf{T}^0(t_0 + h_1 + \cdots + h_n) &= \boldsymbol{\Phi}^{(n)}(h_n)\dots\boldsymbol{\Phi}^{(1)}(h_1)\,\mathbf{T}^0(t_0) \\
&+ \int_0^{h_n} \boldsymbol{\Phi}^{(n)}(h_n - s)\,\mathbf{g}(s + t_0 + h_1 + \cdots + h_{n-1})\,ds \\
&+ \boldsymbol{\Phi}^{(n)}(h_n)\int_0^{h_{n-1}} \boldsymbol{\Phi}^{(n-1)}(h_{n-1} - s)\,\mathbf{g}(s + t_0 + h_1 + \cdots + h_{n-2})\,ds \\
&+ \boldsymbol{\Phi}^{(n)}(h_n)\,\boldsymbol{\Phi}^{(n-1)}(h_{n-1}) \\
&\quad \times \int_0^{h_{n-2}} \boldsymbol{\Phi}^{(n-2)}(h_{n-2} - s)\,\mathbf{g}(s + t_0 + h_1 + \cdots + h_{n-3})\,ds \\
&+ \cdots + \boldsymbol{\Phi}^{(n)}(h_n)\dots\boldsymbol{\Phi}^{(2)}(h_2)\int_0^{h_1} \boldsymbol{\Phi}^{(1)}(h_1 - s)\,\mathbf{g}(s + t_0)\,ds\,.
\end{aligned}
\tag{4.10}
$$

Let us now examine the solution of equation (4.1). We will first prove that if the driving terms are periodic, then the $T_i(t)$ are also (asymptotically) periodic, with the same period of the driving terms.

The solutions of equation (4.1) are given by equations (4.4) and (4.9), from which we obtain in component form $(i = 1, \dots, n)$:

$$T_i(\mu_i t) = \int\limits_0^h \sum_{j=1}^n \boldsymbol{\Phi}_{ij}(h-s)\,g_j[\mu_j(s+t_0)]\,ds + \sum_{j=1}^n \boldsymbol{\Phi}_{ij}(h)\,T_j(\mu_j t_0)\,. \tag{4.11}$$

Since $\mathbf{A}$ is a stability matrix, the last term in equation (4.11) is an exponentially decaying transient; therefore the asymptotic solution, taking into account equation (4.7), is

$$T_i(\mu_i t) = \int\limits_0^h \sum_{m,j=1}^n \bar{\Lambda}_{im}\,\Lambda_{mj}\,e^{\lambda_m(h-s)}\,g_j[\mu_j(t_0 - s)]\,ds\,, \tag{4.12}$$

where $\bar{\Lambda}_{im}$ are the matrix elements of $\boldsymbol{\Lambda}^{-1}$. From now on we will consider periodic driving terms $g_j(t)$ with period τ.

Expanding $g_j(\mu_j t)$ $(j = 1, \dots, n)$ in Fourier series (i denotes the imaginary unit and $\omega_\ell \equiv 2\pi\ell/\tau$):

$$g_j(\mu_j t) = \sum_{\ell=-\infty}^{\infty} c_\ell^{(j)}\,e^{i\omega_\ell t} \tag{4.13}$$

and substituting into equation (4.12), we obtain after integration

$$T_i(\mu_i t) = \sum_{m,j,\ell} \bar{\Lambda}_{im}\,\Lambda_{mj}\,\frac{c_\ell^{(j)}\,e^{i\omega_\ell(t_0 + h)}}{-\lambda_m + i\omega_\ell}\,, \tag{4.14}$$

having further neglected damped exponentials. We can also write

$$T_i(\mu_i t) = \sum_{m,j} \bar{\Lambda}_{im} \Lambda_{mj} \Gamma_m^{(j)}(t_0 + h), \tag{4.15}$$

where

$$\Gamma_m^{(j)}(t_0 + h) \equiv \sum_\ell \frac{c_\ell^{(j)} e^{\,i\omega_\ell\,(t_0+h)}}{-\lambda_m + i\,\omega_\ell}. \tag{4.16}$$

Thus, the $\Gamma_m^{(j)}(t)$ are periodic functions with the same period (pulsation) of the driving terms $g_j(t)$. We are finally able to prove the equivalence between the trilinear and bilinear systems. To this purpose let us write the set of governing differential equations in the case the general n compartment reduction:

$$\begin{cases} \dot{T}_1 = a_{11}\,T_1(t) + \cdots + a_{1n}\,T_n(t) + g_1(t) \\[4pt] \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\[4pt] \dot{T}_n = a_{n1}\,T_1(t) + \ldots + [a_{nn} - \beta]\,T_n(t) + \beta\,T_{\mathrm{E}}(t) + \hat{g}_n(t) \end{cases} \tag{4.17}$$

We point out the parameter β, which is the analog of $\tilde{\beta}_{3\mathrm{E}}$ in the four-compartment reduction; β appears in the heat flux between T_n and T_{E}. If β is very large we expect that the difference between the bilinear and trilinear systems disappears or in other words that

$$\lim_{\beta\to\infty} T_n(t) = T_{\mathrm{E}}(t). \tag{4.18}$$

If we neglect as in the foregoing calculations all decaying exponentials, we obtain the following explicit expression of $T_n(t)$:

$$T_n(\mu_n t) = \sum_{m,j,\ell} \bar{\Lambda}_{nm} \Lambda_{mj}\,(\beta c_\ell^{(j)} + \hat{c}_\ell^{(j)})\,\frac{e^{\,i\omega_\ell\,(t_0+h)}}{-\lambda_m + i\,\omega_\ell}, \tag{4.19}$$

where $c_\ell^{(j)}$ and $\hat{c}_\ell^{(j)}$ are respectively the Fourier coefficients of $T_{\mathrm{E}}(t)$ and $\hat{g}_n(t)$.

To verify equation (4.18), we need to find the β-dependence of Λ_{mj}, of $\bar{\Lambda}_{mj}$, and of the eigenvalues. The problem is solved in the appendix. The result is

$$\lim_{\beta\to\infty} \Lambda_{nm} = \lim_{\beta\to\infty} \Lambda_{mn} = \lim_{\beta\to\infty} \bar{\Lambda}_{nm} = \lim_{\beta\to\infty} \bar{\Lambda}_{mn} = \delta_{mn}, \tag{4.20}$$

where δ_{mn} is the Kronecker delta. Moreover, $-\lambda_n$ diverges linearly in β. Taking into account equations (4.19) and (4.20) we finally arrive at

$$\lim_{\beta\to\infty} T_n(\mu_n t) = \lim_{\beta\to\infty} \sum_\ell c_\ell^{(n)} e^{\,i\omega_\ell\,(t_0+h)} = T_{\mathrm{E}}[\mu_n(t_0 + h)], \tag{4.21}$$
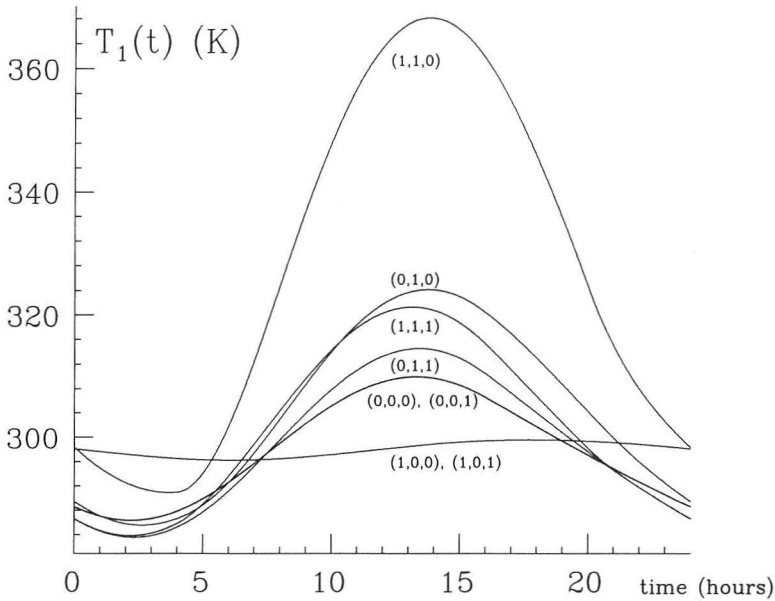
from which we deduce equation (4.18).

Figure 1: The time evolution over a period τ of the core temperature of the eight uncontrolled dynamical systems.

5. The bilinear system $Z(4,3)$ with constant controls

A particular case of the system $Z(4,3)$ is obtained by considering each coefficient c_{ij} simply as a constant parameter, equal to either 0 or 1.

We refer for the nomenclature to [1], section 5. The constant controls are obtained formally by assigning an arbitrary initial value to the control vector $\mathbf{c}$, say $\mathbf{c}(t_0) = \mathbf{C}$; in fact, imposing an infinitely wide floating interval η (see [1], section 5.2), the given initial triplet $\mathbf{C}$ shall be preserved throughout ($\mathcal{N}^{\text{sw}} = 0$). The recipe is therefore

$$\begin{cases} \text{floating interval } \eta \to \infty, \\ c_{ij}(t_0) = C_{ij} \quad (ij = 12, 23, 3\text{E}). \end{cases} \tag{5.1}$$

We recall from [1], section 5.1, that by varying continuously the set point value T_{b} we generate an infinite family of controlled systems $Z(4,3)$ whose motion is $\mathbf{T}(t)$.

Let the constant-parameter motion be indicated by $\mathbf{T^C}(t)$; in correspondence to the 2^3 possible constant controls $\mathbf{C} \equiv (0,0,0)$, $(0,0,1)$, ..., $(1,1,1)$ there are 8 *uncontrolled* dynamical systems whose motion is respectively $\mathbf{T}^{(0,0,0)}(t)$, $\mathbf{T}^{(0,0,1)}(t)$, ..., $\mathbf{T}^{(1,1,1)}(t)$. Figure 1 shows the evolution of $T_1^{(0,0,0)}(t)$, $T_1^{(0,0,1)}(t)$, ..., $T_1^{(1,1,1)}(t)$.
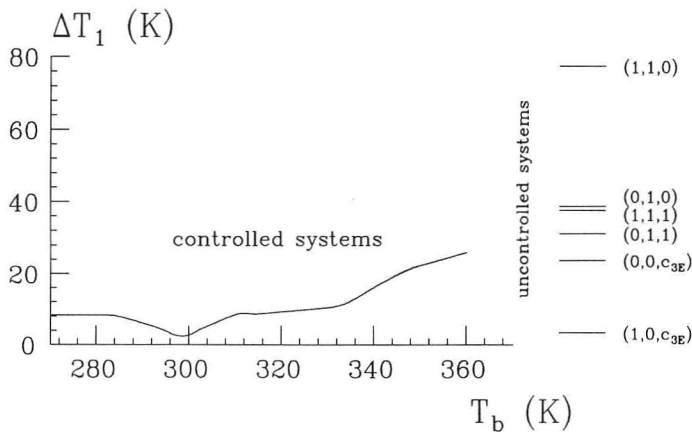
Figure 2: The dependence of the daily oscillation ΔT_1 corresponding to the eight uncontrolled dynamical systems (notice that some levels are degenerate), compared with the ΔT_1 of $Z(4,3)$, which depends on the set point T_b.

controls	ΔT_1 (K)	δ	$\langle T_1 \rangle$ (K)	ϑ (K)
$(0,0,0)$	23.66	0.78	298.08	0.06
$(0,0,1)$	23.67	0.78	298.03	0.02
$(0,1,0)$	38.68	1.28	304.27	6.25
$(0,1,1)$	31.03	1.03	298.81	0.80
$(1,0,0)$	3.36	0.11	298.05	0.04
$(1,0,1)$	3.36	0.11	298.05	0.04
$(1,1,0)$	77.34	2.56	327.41	29.40
$(1,1,1)$	37.54	1.24	301.82	3.81

Table 1:

Figure 2 shows the daily oscillation of the core temperature ΔT_1 of the uncontrolled systems compared with the ΔT_1 of the controlled systems (continuous distribution over T_b).

Table 1 gives, in correspondence to each triplet $\mathbf{C}$,

1. the daily average core temperature $\langle T_1 \rangle$;

2. the heating index [2] $\vartheta \equiv \langle T_1 \rangle - \langle T_E \rangle$ (K);

3. the daily oscillation ΔT_1;

4. the stability index [2] $\delta \equiv \Delta T_1 / \Delta T_E$.

Notice the excellent stability index corresponding to $\mathbf{C} = (1, 0, c_{3E})$ with $c_{3E} = 0$ or 1 (constant). In fact, the triplets $(1, 0, c_{3E})$ correspond to strong conduction between the core 1 and the structure 2 ($c_{12} = 1$) and strong

insulation between the structure and 3 ($c_{23} = 0$), which in turn may or may not be strongly coupled to the external temperature ($c_{3E} = 0, 1$).

The well-behaving $T_1^{(1,0,1)}(t)$ would seem to be comparable to the controlled $T_1(t)$. However, the constant-control policy works out only with one specific value of T_b, that is, the mean daily value $\langle T_1^{(1,0,1)} \rangle$. This value is not preassigned, but rather obtained a *posteriori*. With a small variation of T_b from this value, the constant triplet $(1, 0, 1)$ fails to control. We can obtain analytic relations among the daily average values $\langle T_i \rangle$ when the controls are constant and therefore when $\langle c_{ij} T_i \rangle = c_{ij} \langle T_i \rangle$.

It was shown in section 4 that if the driving terms are periodic with period τ, then each $T_i(t)$ is periodic with the same period and therefore $\langle \dot{T}_i \rangle = T_i(t) - T_i(t - \tau) = 0$. Hence, integrating over a daily period τ the trilinear equations (2.1) with constant controls we find

$$\langle T_1 \rangle = \frac{(\beta_{12} + c_{12} \tilde{\beta}_{12}) \langle T_2 \rangle + \beta_{1E} \langle T_E \rangle}{\beta_{12} + c_{12} \tilde{\beta}_{12} + \beta_{1E}}$$

$$\langle T_2 \rangle = \frac{(\beta_{12} + c_{12} \tilde{\beta}_{12}) \langle T_1 \rangle + (\beta_{23} + c_{23} \tilde{\beta}_{23}) \left[(1 - c_{3E}) \langle T_3 \rangle + c_{3E} \langle T_E \rangle \right]}{\beta_{12} + c_{12} \tilde{\beta}_{12} + \beta_{23} + c_{23} \tilde{\beta}_{23}}$$

$$(1 - c_{3E}) \langle T_3 \rangle = (1 - c_{3E}) \frac{(\beta_{23} + c_{23} \tilde{\beta}_{23}) \langle T_2 \rangle + \beta_{34} \langle T_4 \rangle + \beta_{3E} \langle T_E \rangle}{\beta_{23} + c_{23} \tilde{\beta}_{23} + \beta_{34} + \beta_{3E}}$$

$$\langle T_4 \rangle = (1 - c_{3E}) \langle T_3 \rangle + c_{3E} \langle T_E \rangle + \frac{S_{4E}}{\beta_{34}} \langle \Delta \phi \rangle. \qquad (5.2)$$

The term $\langle T_E \rangle$ is assumed to be known or easily measurable. The term $\langle \Delta \phi \rangle$ instead is not a given input but depends on the system dynamics, since it represents the balance of the radiation flux (in W m^{-2}) over τ, which is $\langle \phi^{in} \rangle - \langle \phi^{out} \rangle$, ϕ^{in} being a driving term while the term ϕ^{out} contains the nonlinearity T_4^4 (see equation (2.1)). Similar relations may be found by integrating the bilinear equations with constant controls.

As a final remark on the uncontrolled systems, notice in figure 3 that rather unexpectedly the entropy average production $\langle \Sigma \rangle$ (see [1], section 6) of some uncontrolled configurations is smaller than the controlled $\langle \Sigma \rangle (T_b)$.

6. Conclusion

In section 4 the bilinear model was proved to contain the trilinear model. The numerical equivalence between the two systems in the limit $\tilde{\beta}_{3E} \to \infty$ may be shown in the following way.

A characteristic parameter of $Z(4, 3)$ is the maximum deviation Δ_{3E} of $T_3(t)$ from $T_E(t)$ over τ; figure 4 shows Δ_{3E} as a function of $\tilde{\beta}_{3E}$ when $\mathbf{c} = (1, 0, 1)$. We see that Δ_{3E} approaches zero as $\tilde{\beta}_{3E} \to \infty$, and therefore the bilinear system approaches the trilinear system.
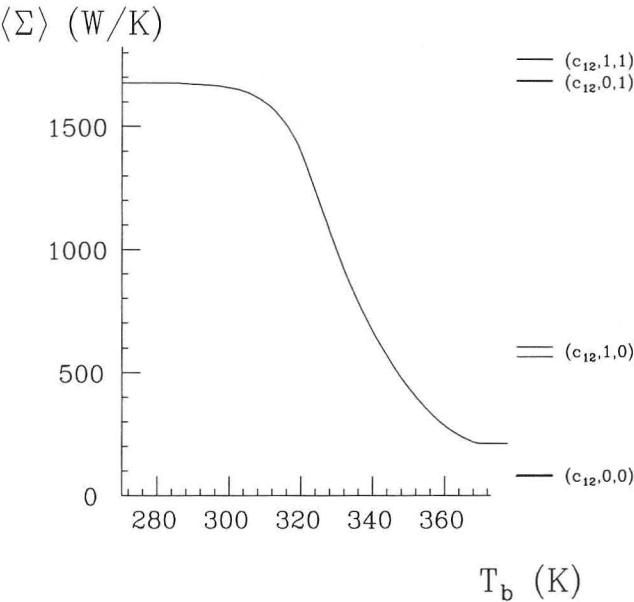
Figure 3: The dependence on the set point T_b of the average entropy production $\langle \Sigma \rangle$ of $Z(4,3)$, compared with the levels corresponding to the eight uncontrolled dynamical systems. Notice the degeneracy of the uncontrolled levels.
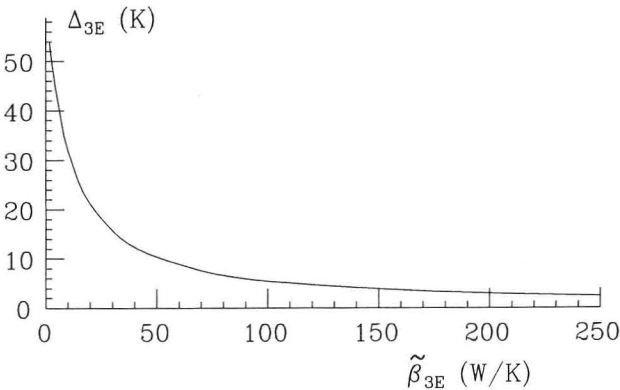

Figure 4: The dependence of the maximum deviation Δ_{3E} of $T_3(t)$ from $T_E(t)$ over a period, as a function of the parameter $\tilde{\beta}_{3E}$. Thus we see that the bilinear system of equations describes the same model as the trilinear system when $\tilde{\beta}_{3E}$ is large enough.

Appendix

The following matrix $\mathbf{A}$ was introduced in section 4:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} - \beta \end{pmatrix}. \tag{A.1}$$

In particular we are concerned with the eigenvalues λ_i and with the modal matrix Λ of $\mathbf{A}$.

A.1   The eigenvalues λ_i

The eigenvalues of $\mathbf{A}$ are subject to the following constraints:

1. $\lambda_i < 0$. $\lim_{\beta \to \infty} \lambda_i$ cannot be positive or 0.

2. $\mathrm{Tr}\,\mathbf{A}$ is linear in β.

3. $\det \mathbf{A}$ is linear in β.

Therefore only one of the λ_i can diverge in the limit $\beta \to \infty$ and the divergence is linear in β. For symmetry reasons the diverging eigenvalue is λ_n:

$$\lambda_n \sim -\beta \quad \text{when } \beta \text{ is large}. \tag{A.2}$$

A.2   The matrix Λ

The evaluation of the matrix elements Λ_{ij} and $\bar{\Lambda}_{ij}$ reduces to the determination of the eigenvectors of $\mathbf{A}$. In fact the components of the eigenvectors appear on the rows of Λ and on the columns of Λ^{-1}. The standard evaluation of eigenvectors yields:

$$\Lambda_{i\ell} = (-1)^{n-\ell} \frac{\Lambda_{in}}{D} A_{n\ell}, \tag{A.3}$$

$$\begin{aligned} \Lambda_{i\ell} = \frac{\Lambda_{in}}{a_{n\ell}} \Big[&\beta + \lambda_i - a_{nn} - D^{-1} \left((-1)^{n-1} A_{n1}\, a_{n1} + \cdots \right. \\ &+ (-1)^{n-\ell+1} A_{n,\ell-1}\, a_{n,\ell-1} + (-1)^{n-\ell-1} A_{n,\ell+1}\, a_{n,\ell+1} \\ &+ \cdots - A_{n,n-1}\, a_{n,n-1} \Big], \end{aligned} \tag{A.4}$$

where $D(\lambda_i)$ is the determinant of the principal minor of order $n - 1$, and $A_{n\ell}(\lambda_i)$ is the determinant of the algebraic complement of $a_{n\ell}$. Both expressions are polynomials in λ_i of degree $n - 1$ and $n - 2$, respectively.

Remarks:

1. $\Lambda_{\ell n}$ cannot diverge since we chose normalized eigenvectors.

2. $\lambda_i \, (i \neq n)$ does not diverge in β.

As a consequence, when $i \neq n$ the β-divergence on the second side of equation (A.4) must be cancelled. We have:

$$\Lambda_{in} \sim \frac{A_i}{B_i + C_i \, \beta}, \qquad \lim_{\beta \to \infty} \Lambda_{in} = 0 \quad (i \neq n). \tag{A.5}$$

In the case $i = n$, the expression within brackets in equation (A.4) does not diverge and as a consequence $\lim_{\beta \to \infty} \Lambda_{in}$ must be finite and different from 0. From equation (A.3) we deduce that

$$\Lambda_{n\ell} \sim \frac{1}{\beta} \tag{A.6}$$

and

$$\lim_{\beta \to \infty} \Lambda_{n\ell} = 0. \tag{A.7}$$

Taking into account the normalization condition we have

$$\lim_{\beta \to \infty} \Lambda_{nn} = 1. \tag{A.8}$$

Remarks:

1. We did not use equation (A.4) to deduce equation (A.6) because our present knowledge does not allow us to verify that the asymptotic behavior of λ_n cancels also the term a_{nn}.

2. The normalization conditions are also compatible with the value -1 in equation (A.8) but the normalized eigenvalues are always defined up to an overall constant of modulus one.

Acknowledgments

References

[1] G. Cosenza, A. Massobrio, and L. Sertorio, "A Thermodynamic Automaton with Four Temperatures and Three Controls," *Complex Systems*, **5** (1991) 541–562.

[2] A. Massobrio and L. Sertorio, "A Theoretical Approach to Solar Architecture," *Nuovo Cimento B*, **106** (1991) 69–90.