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Dynamical Behavior of a Neural Automaton with Memory

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Abstract. We study the dynamics of an automaton with memory whose equation is the following:

$$x_{n+1} = 1[\sum_{i=0}^{k-1} a_i x_{n-1} - \Theta]$$

where $\mathbf{a} = (a_i)_{i=0\ldots k-1}$ denotes the coupling coefficients vector. We show that if $\mathbf{a}$ is symmetric, then we can introduce an energy operator; thereby we state that the periods of the automaton always divide $(k+1)$ and give a bound of the transient. We also study the case of reversible systems and characterize reversibility versus the coupling coefficients. Thereafter, we give some results about the pivot sums systems. Some conjectures concerning the general case are given.

1. Introduction

In this paper, we study the dynamical behavior of an automaton with a bounded memory. The equation of the automaton is the following:

$$x_{n+1} = 1[\sum_{i=0}^{k-1} a_i x_{n-i} - \Theta] \tag{1.1}$$

where

x_n is the state of the automaton at the discrete time step n; it is a Boolean variable;

a_i are the coupling coefficients; they are real constants;

Θ is the threshold; it is a real parameter;

k is the size of the memory; it is an integer constant;

$1[u] = 0$ if $u < 0$ and $1[u] = 1$ if $u \geq 0$.

This automaton has been essentially introduced in order to modelize elementary electrical properties of the nervous system [1-3,13]. Equation (1.1) is called a single neuronic equation and modelizes the behavior of a single neuron. In this case, the resting state is represented by "0", and the firing state by "1".

The general dynamic of such a model is extremely rich. Indeed, it is shown in [13] that if we connect various automata of this kind–called formalized neurons–we can simulate every finite automaton.

This model has been studied by many authors [1-5,12,14,15] for some particular choices of the coupling coefficients a_i and the threshold Θ. In this paper, we present some results concerning these particular choices and we give some conjectures concerning the general case. More precisely, we study the case of palindromic systems and characterize the reversible automata according to the values of a_i and Θ. We also study the case of the pivot-sum systems.

Clearly, equation (1.1) is completely defined by the pair (a, Θ) where $\mathbf{a} = (a_0, a_1, \ldots, a_{k-1})$. It can be transformed into a system of order k in the classical following ways

$$
\begin{aligned}
&T : \{0,1\}^k \rightarrow \{0,1\}^k \\
&y(n) = (y_1(n), y_2(n), \ldots, y_k(n)) \mapsto y(n+1) = T(y(n)) \qquad (1.2) \\
&= (y_2(n), y_3(n), \ldots, y_k(n), f(y(n)))
\end{aligned}
$$

where

$$f(y(n)) = 1[\sum_{i=0}^{k-1} a_i\, y_{k-i}(n) - \Theta]$$

Hence, the automaton is equivalent to a discrete iteration on $\{0,1\}^k$. Using this equivalence, (1.1) can also be seen as an automata network. Indeed, let $T_1, \ldots, T_k$ be the components of $T : y_i(n+1) = T_i(y(n))$. Clearly, $y_k(n+1) = T_k(y(n)) = f(y(n))$, and for $1 \leq i \leq k-1$, $y_i(n+1) = T_i(y(n)) = y_{i+1}(n) = 1[y_{i+1}(n) - 1/2]$.

The network is composed of k threshold automata (without memory). Henceforth, the study of (1.1) can be included in the general framework of the finite networks of threshold automata:

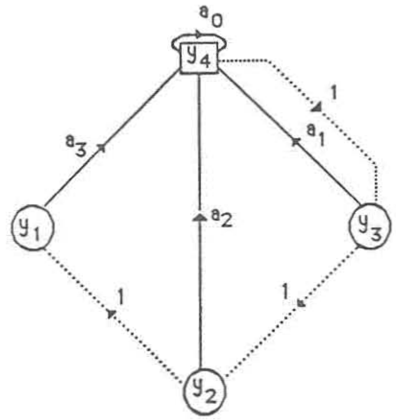

Figure 1: Directed graph associated to a single neuronic equation with memory of length $k = 4$ and coupling coefficient vector $\mathbf{a} = (a_0, a_1, a_2, a_3)$. Circles: the state of the cell i is the previous state of its neighbor cell $i+1$. Squares: the state of the cell 4 is computed by taking into account all the cells of its neighborhood cells 1, 2, 3, and 4; with a threshold rule f.

$$y_i(n+1) = 1[\sum_{j=1}^{k} m_{ij}\, y_j(n) - \Theta_i] \tag{1.3}$$

where $M = (m_{ij})_{1 \le i,j \le k}$ is a real matrix and $(\Theta_i)_{i=1,\dots,k}$ is a real k-vector. Figure 1 shows the graph associated to the neuronic equation.

In [8], Eric Goles studies the case of a real symmetric matrix M. He proves that, in this case, the period of the cycles of (1.3) is less than or equal to 2.

On the other hand, it is easily seen that the previous system (1.2) can be written as follows:

$$y_i(n+1) = 1[\sum_{j=1}^{k} m_{ij}\, y_j(n) - \Theta_i]$$

where $M = (m_{ij})_{1 \le i,j \le k}$ is a real matrix defined by

$$m_{ij} = \begin{cases} 1 & \text{if } j - i = 1 \text{ and } i < k \\ a_{i-j} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Note that in the former example, the matrix M is not symmetric. Very few results are known in this case.

Moreover, to each arc $(i+1,i)_{i=1,\dots,k}$ we associate a weight equal to 1, and to each arc $(i,k)_{i=1,\dots,k}$ we associate a weight equal to a_{k-i}.

In the following, let $A(\mathbf{a},\Theta)$ denote the automaton defined by the equation (1.1).

> We say that two automata are equivalent if they have the same iteration graph.
>
> $A(\mathbf{a},\Theta)$ is a strict threshold automaton if for every $\mathbf{x}=(x_0, x_1, x_2, \dots, x_{k-1}) \in \{0,1\}^k$, we have
>
> $$\sum_{i=0}^{k-1} a_i x_i - \Theta \neq 0$$

Note that for every automaton $A(\mathbf{a},\Theta)$ there always exists a real number ε such that $A(\mathbf{a},\Theta+\varepsilon)$ is a strict threshold automaton and $A(\mathbf{a},\Theta) \equiv A(\mathbf{a},\Theta+\varepsilon)$.

Hence, without lost of generality, we can always assume that $A(\mathbf{a},\Theta)$ is a strict threshold automaton.

Lemma 1. *Let $A(\mathbf{a},\Theta)$ be a given strict threshold automaton. Then there always exists an equivalent strict threshold automaton $A(\mathbf{a}^*,\Theta^*)$ such that the coefficients a_i^* are integers.*

Proof. If the coefficients are rational $a_i = m_i/d_i$ then let $a_i^* = M.a_i$ and $\Theta^* = M \times \Theta$ where $M = lcm(d_i)$.

If there exists an irrational coefficient a_i then we can always find rational numbers a_i^* such that $a_i < a_i^* \le a_i + \varepsilon/2k$ where $\varepsilon = \inf\{|\sum a_i x_i - \Theta|$ where $x_i \in \{0,1\}\}$. Clearly $\varepsilon > 0$ because we have assumed that $A(\mathbf{a},\Theta)$ is a strict threshold automaton. We can easily verify that $A(\mathbf{a},\Theta)$ is equivalent to $A(\mathbf{a}^*,\Theta)$: since $a_i \le a_i^*$, we have

$$\text{if } \sum a_i x_i - \Theta \ge 0 \text{ then } \sum a_i^* x_i - \Theta \ge 0$$

If $\sum a_i x_i - \Theta < 0$ then we have $\sum a_i x_i - \Theta < -\varepsilon$, which implies that $\sum a_i^* x_i - \Theta < -\varepsilon/2 < 0$ and concludes the proof. ∎

From now on we shall assume that the coefficients a_i are integers.

2. Symmetric memory

We assume in this section that the coupling coefficients of (1.1) form a symmetric word; that is, $a_i = a_{k-1-i}$ for every $i \in \{0,\dots,k-1\}$.

Following E. Goles [8], we can introduce an operator E characterizing the dynamical behavior of such systems: for each $\mathbf{x} = (x_0, x_1, \dots, x_k) \in \{0,1\}^{k+1}$, we define the operator E by

$$E(x) = -\sum_{j=0}^{k-1} x_{k-j} \sum_{s=j+1}^{k} a_{s-j-1} x_{k-s} + \Theta \sum_{j=0}^{k} x_j$$

More generally for a trajectory $(x_i)_{i\in N}$ we have

$$E(x_{n-k}, x_{n-k+1}, \dots, x_n) = -\sum_{j=0}^{k-1} x_{n-j} \sum_{s=j+1}^{k} a_{s-j-1} x_{n-s} + \Theta \sum_{j=0}^{k} x_{n-j}$$

The operator E can be seen as a kind of energy of the system. Indeed, E is a Lyapunov function. (For a more general approach, see E. Goles [8].) The variation of E gives an idea of how the system reaches a stationary state. For this purpose, we define a quantity Δ_n by

$$\Delta_n = E(x_{n-k}, \dots, x_n) - E(x_{n-k-1}, \dots, x_{n-1})$$

Let $(x_i)_{i\in N}$ be a trajectory of the system whose period and transient will be denoted by p and q respectively. Since the coupling coefficients a_i present a symmetric structure, we get

$$\begin{aligned}\Delta_n &= E(x_{n-k}, \dots, x_n) - E(x_{n-k-1}, \dots, x_{n-1}) \\ &= -(x_n - x_{n-k-1})(\sum_{s=1}^{k} a_s x_{n-s} - \Theta)\end{aligned}$$

Lemma 2. *If $x_n \neq x_{n-k-1}$ then $\Delta_n < 0$.*

Proof. Suppose that $x_n \neq x_{n-k-1}$. If $x_n = 1$ (which implies that $x_{n-k-1} = 0$) then

$$\sum_{s=0}^{k-1} a_s x_{n-s-1} - \Theta > 0$$

If $x_n = 0$ then the proof is the same. ∎

Theorem 1. *The period p of a cycle of a given symmetric memory system $A(\mathbf{a}, \Theta)$ divides $k+1$.*

Proof. Note that for every i we have $\Delta_i \leq 0$. Since $x_{q+i} = x_{q+p+i}$ for $i = 0, 1, 2, \dots, k$ we deduce that

$$\sum_{i=0}^{p} \Delta_{q+k+i} = E(x_q, x_{q+1}, \dots, x_{q+k}) - E(x_{q+p}, x_{q+p+1}, \dots, x_{q+k+p})$$

$$= 0$$

Hence $\Delta_{q+k+i} = 0$ for every $i = 0, 1, 2, \dots, p$. Lemma 2 implies that $x_{k+q+i} = x_{q+i-1}$ for every $i = 1, 2, \dots, p$ which proves that p divides $k+1$. ∎

The preceding operator can be used in order to derive a general bound for the length of the transient.

Theorem 2. *The transient q of a given symmetric memory system $A(\mathbf{a}, \Theta)$ is bounded by*

$$q \leq (k+1)^2[2|\Theta| + \sum_{s=0}^{k-1} |a_s|]$$

Proof. It is easily seen that for every $\mathbf{y} = (y_0, y_1, \ldots, y_k) \in \{0,1\}^{k+1}$ we have

$$-\sum_{j=0}^{k-1} \sum_{\substack{s=j+1 \\ a_s>0}}^{k} a_{s-j-1} \leq E(y) \leq \Theta(k+1)$$

$$-\sum_{j=0}^{k-1} \sum_{\substack{s=j+1 \\ a_s<0}}^{k} a_{s-j-1} \qquad \text{if } \Theta \geq 0$$

and

$$-\sum_{j=0}^{k-1} \sum_{\substack{s=j+1 \\ a_s>0}}^{k} a_{s-j-1} + \Theta(k+1) \leq E(y) \leq$$

$$-\sum_{j=0}^{k-1} \sum_{\substack{s=j+1 \\ a_s<0}}^{k} a_{s-j-1} \qquad \text{if } \Theta \leq 0$$

By applying this formula respectively to the vectors $(x_q, x_{q+1}, \ldots, x_{q+k})$ and $(x_0, x_1, \ldots, x_k)$ we obtain

$$|E(x_q, x_{q+1}, \ldots, x_{q+k}) - E(x_0, x_1, \ldots, x_k)| \leq (k+1)|\Theta|$$

$$+\sum_{j=0}^{k-1} \sum_{s=j+1}^{k} |a_{s-j-1}|$$

Since $\sum_{j=0}^{k-1} \sum_{s=j+1}^{k} |a_{s-j-1}| = \frac{(k+1)}{2} \sum_{i=0}^{k-1} |a_i|$ we can write

$$|E(x_q, x_{q+1}, \ldots, x_{q+k}) - E(x_0, x_1, \ldots, x_k) \leq (k+1)|\Theta|$$

$$+\frac{(k+1)}{2} \sum_{s=0}^{k-1} |a_s|$$

Applying lemma 1, we deduce that

$$E(x_q, x_{q+1}, \ldots, x_{q+k}) - E(x_0, x_1, \ldots, x_k) = \sum_{i=1}^{q} \Delta_{k+i} < 0$$

Let us call $d = \min\{(-\Delta_{k+1})$ such that $\Delta_{k+1} < 0$ and $i = 1, 2, \ldots, q\}$. On the other hand, we have

$$\begin{aligned} d &= \min_{\substack{i = 1,2,\ldots,q \\ x_{i-1} \neq x_{k+i}}} [|(x_{i-1} - x_{k+i})|.|\sum_{s=0}^{k-1} a_s\, x_{k+i+s-1} - \Theta|] \\ &= \min_{\substack{i = 1,2,\ldots,q \\ \Delta_{k+i} \neq 0}} |\sum_{s=0}^{k-1} a_s\, x_{k+i+s-1} - \Theta| \text{ since } |x_{k+i} - x_i| = 1. \end{aligned}$$

As the coefficients a_i are integers, we can easily verify that $d \geq 1/2$. Let $q = \alpha + \beta$ with

$$\alpha = \text{card } \{i \leq q \text{ such that } \Delta_{k+i} = 0\}$$

$$\beta = \text{card } \{i \leq q \text{ such that } \Delta_{k+i} \neq 0\}$$

Clearly, we have $\alpha \leq k\beta$ since $(\Delta_{k+i})_{0 \leq i \leq q}$ cannot contain $(k+1)$ successive zeros. So we get

$$d \times \beta \leq (k+1)[|\Theta| + \frac{1}{2}\sum_{s=0}^{k-1} |a_s|]$$

with $d \geq 1/2$.

As $q = \alpha + \beta$ we can write $q \leq (k+1)\beta$, which implies that

$$q \leq (k+1)^2[2|\Theta| + \sum_{s=0}^{k-1} |a_s|]$$

∎

The preceding analysis can be extended to the following case:

$$x_{n+1} = 1[\sum_{i=r}^{k-1} a_i x_{n-i} - \Theta] \tag{2.1}$$

where the coefficients a_i are such that $a_i = a_{k+r-i-1}$ for $i \in \{r, r+1, \ldots, k-1\}$.

Lemma 3. *The automaton defined by (2.1) is equivalent to the following*

$$x_{n+1} = 1[\sum_{i=0}^{k+r-1} a'_i x_{n-i} - \Theta] \tag{2.2}$$

with

$$a'_i = \begin{cases} a_i & \text{if } i = r, r+1, \ldots, k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1. *If $(x_i)_{i \in N}$ is a trajectory of the system (2.1) with a transient length q and a period p, then we have*

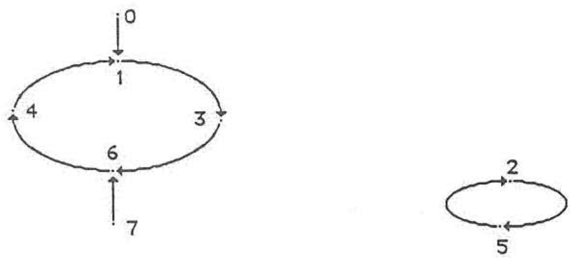

Figure 2: Iteration graph associated with example 2.

the period p always divides $(k+r+1)$,

the transient q is bounded by

$$q \leq (k+r+1)^2[2|\Theta| + \sum_{i=r}^{k-1} |a_i|]$$

Proof. Follows directly from theorems 1 and 2. ■

Example 1. *Let $k = 3$, $\Theta = -2.5$ and $\mathbf{a} = (-2, -1, -2)$*

$$x_{n+1} = 1[2,5 - (2x_n + x_{n-1} + 2x_{n-2})]$$

We can see on figure 2 that $A(\mathbf{a}, t)$ possesses two cycles and that the period of each cycle divides $(k+1)$.

Corollary 2. *Let $A(\mathbf{a}, \Theta)$ be a given symmetric memory system whose coefficients are such that*

$a_i \in \{-1, 0, 1\}$ *then* $q \leq 3k(k+1)$

(polynomial bound versus exponential possible states).

Proof. Obvious. ■

We now consider a particular case of the symmetric memory. We shall assume that the coupling coefficients are of the form $a_{2i} = 1$ and $a_{2i+1} = 0$.

Proposition 1. *If $\Theta \geq \lceil k/2 \rceil$ or $\Theta \leq 0$ then $A(\mathbf{a}, \Theta)$ has only fixed points 0 or 1 and no cycle of order greater than 1.*

If $0 < \Theta < \lceil k/2 \rceil$ then $A(\mathbf{a}, \Theta)$ has a unique globally attractive cycle of order 2 : $(0, 1)$.

Proof. The first part of the proposition is obvious since the sign of $\Theta + \sum a_i x_{n-i}$ is always the same.

In order to prove the second part, remark that since $a_{2i+1} = 0$, we can assume that k is odd. Let $(x_0, x_1, \ldots, x_{k-1})$ be an initial configuration and let C be its associated cycle of period p. To prove that C is reduced to $(0,1)$, we shall show that C has neither two consecutive 1 nor two consecutive 0.

Since k is odd, we have

$$x_{n+1} = 1[(x_n + x_{n-2} + x_{n-4} + \ldots + x_{n-k+1}) - \Theta] \tag{2.3}$$

Let us assume that C has two consecutive 1: there exists n such that $x_{n-k-1} = x_{n-k} = 1$ with $n - k - 1 \geq q$, where q denotes the length of the transient.

Note that the vector $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_{k-1})$ is symmetric, which implies that p divides $(k+1)$. Thus, $x_n = x_{n-k-1} = 1$ and $x_{n+1} = x_{n-k} = 1$. On the other hand, we have

$$\begin{aligned} x_{n+2} &= 1[(x_{n+1} + x_{n-1} + \ldots + x_{n-k+2}) - \Theta] \\ &= 1[(x_{n-1} + x_{n-3} + \ldots + x_{n-k+2} + x_{n+1}) - \Theta] \\ &= 1[(x_{n-1} + x_{n-3} + \ldots + x_{n-k+2} + x_{n-k}) - \Theta] \\ &= x_n \end{aligned}$$

which proves that $x_n = x_{n+1} = x_{n+2} = 1$. With the same reasoning, we get $X_{n+r} = 1$. Hence $C = (1, 1, \ldots, 1)$ which gives a contradiction to the hypothesis $\Theta < \lceil k/2 \rceil$.

If we assume that C has two consecutive 0, we can easily show in a similar way that $x_{n+r} = 0$ for any integer $r \geq 1$, which leads to the same conclusion. ∎

Corollary 3. *If* $0 < \Theta$ *then* $q \leq 3(k+1)^2 \lceil k/2 \rceil$.

If $\Theta \leq 0$ *then* $q \leq (k+1)^2 (1 + \lceil k/2 \rceil)$.

Proof. A direct consequence of theorem 2. ∎

3. Positive coupling coefficients

Note that if the coupling coefficients a_i are positive, then the operator T defined as follows is a monotone map:

$$T : \mathrm{x}_n = (x_{n-k+1}, x_{n-k+2}, \ldots, x_n) \to (x_{n-k+2}, x_{n-k+3}, \ldots, x_n, f(\mathrm{x}_n))$$

where

$$f(x_{n-k+1}, x_{n-k+2}, \ldots, x_n) = 1[\sum_{i=0}^{k-1} a_i x_{n-i} - \Theta]$$

Hence it is well known that if T generates a p-cycle $\mathbf{x}$, $T(\mathbf{x})$ $T^2(\mathbf{x}),\ldots, T^{p-1}(\mathbf{x})$ (with $p > 1$) then $\mathbf{x}$, $T(\mathbf{x})$, $T^2(\mathbf{x}),\ldots,T^{p-1}(\mathbf{x})$ are uncomparable relatively to the partial ordering. If there exist two integers n and m such that $\mathbf{x}_n = (x_{n-k+1}, x_{n-k+2},\ldots,x_n)$ and $\mathbf{x}_m = (x_{m-k+1}, x_{m-k+2},\ldots,x_m)$ are comparable, then we have a cycle whose length divides $|m-n|$. In particular, if there exists an integer n such that $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are comparable, then the sequence $(x_i)_{i\in N}$ converges toward a fixed point.

Now let us see the particular case when the coupling coefficients are decreasing; i.e.,

$$0 \le a_{k-1} \le a_{k-2} \le \ldots \le a_{j+1} < a_j \le \ldots \le a_0$$

Proposition 2. *If the coupling coefficients are decreasing positive then the automaton $A(\mathbf{a},\Theta)$ has only fixed points.*

Proof. We shall prove that if $x_n = 0$ then $x_{n+1} = 0$.

Let us assume that $x_n = 0$, which is equivalent to $a_0x_{n-1} + a_1x_{n-2} + a_2x_{n-3} + \ldots + a_{k-1}x_{n-k} - \Theta < 0$. Then we have $a_0x_n + a_1x_{n-1} + \ldots + a_{k-1}x_{n-k+1} - \Theta < a_0x_{n-1} + a_1x_{n-2} + \ldots a_{k-2}x_{n-k+1} + a_{k-1}x_{n-k} - \Theta < 0$, which implies that $x_{n+1} = 0$.

Note that if the coupling coefficients are increasing positive, then we have not the same result as before. ∎

Example 2.

$$x_{n+1} = 1[x_n + 2x_{n-1} + 4x_{n-2} + 6x_{n-3} - 5.5] \tag{3.1}$$

As it is shown in figure 3, equation (3.1) admits two cycles of period 4 and 2 and two fixed points.

Remark that the period of each cycle divides k. This is a direct consequence of theorem 4 since we have $a_{k-1} - \Theta \le 0$.

4. Reversible systems

In general, the automaton $A(\mathbf{a},\Theta)$ is not reversible in the sense that from two different initial configurations $(x_{n-k+1}, x_{n-k+2}, \ldots, x_n)$ and $(y_{n-k+1}, y_{n-k+2}, \ldots, y_n)$ we can get the same final state.

Let us associate to the automaton $A(\mathbf{a},\Theta)$ the operator T defined on $\{0,1\}^k$ by

$$\mathbf{x} = (x_0, x_1, \ldots, x_{k-1}) \to T(\mathbf{x}) = (x_1, x_2, \ldots, x_{k-1}, f(\mathbf{x}))$$

where

$$f(x_0, x_1, \ldots, x_{k-1}) = 1[\sum_{k=0}^{k-1} a_i x_{k-1-i} - \Theta]$$

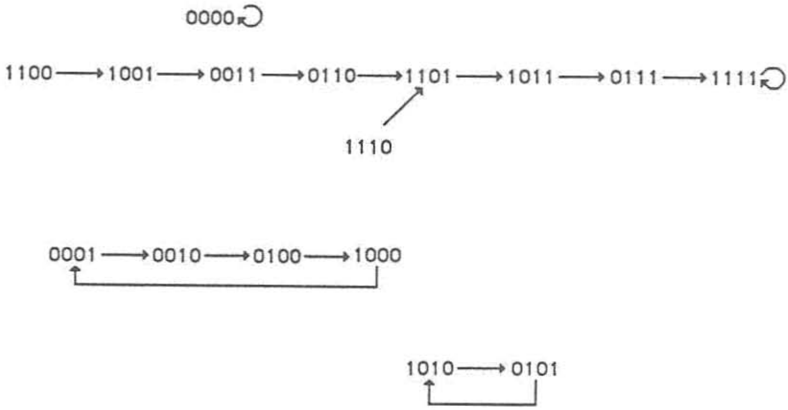

Figure 3: Iteration graph associated with example 2.

Definition 1. *$A(\mathbf{a},\Theta)$ is a reversible automaton if T is a bijection.*

Definition 2. *$A(\mathbf{a},\Theta)$ is a shift if $f(x_0,x_1,\dots,x_{k-1}) = x_0$; that is,*

$$T(x_0,x_1,\dots,x_{k-1}) = (x_1,x_2,\dots,x_{k-1},x_0).$$

$A(\mathbf{a},\Theta)$ is an antishift if $f(x_0,x_1,\dots,x_{k-1}) = 1-x_0$.

Clearly, if $A(\mathbf{a},\Theta)$ is a shift, then for every $\mathbf{x}$ in $\{0,1\}^k$ we have $T^k(\mathbf{x}) = \mathbf{x}$, and if $A(\mathbf{a},t)$ is an antishift, then we have $T^{2k}(\mathbf{x}) = \mathbf{x}$. We deduce from this remark that if $A(\mathbf{a},\Theta)$ is a shift or an antishift then $A(\mathbf{a},t)$ is reversible. Moreover, we have $T^{-1} = T^{k-1}$ if $A(\mathbf{a},\Theta)$ is a shift and $T^{-1} = T^{2k-1}$ if $A(\mathbf{a},\Theta)$ is an antishift.

Proposition 3. *$A(\mathbf{a},\Theta)$ is reversible if and only if $A(\mathbf{a},\Theta)$ is a shift or an antishift.*

Proof. We shall prove that if $A(\mathbf{a},\Theta)$ is reversible, then $A(\mathbf{a},\Theta)$ is a shift or an antishift. We distinguish three cases: $a_{k-1} > 0$, $a_{k-1} < 0$, $a_{k-1} = 0$.

If $a_{k-1} > 0$, then we have

$$\sum_{i=0}^{k-2} a_i x_{k-1-i} - \Theta < a_{k-1} + \sum_{i=0}^{k-2} a_i x_{k-1-i} - \Theta$$

which implies that $f(0,x_1,x_2,\dots,x_{k-1}) = 0$ and $f(1,x_1,x_2,\dots,x_{k-1}) = 1$. Hence $A(\mathbf{a},\Theta)$ is a shift.

If $a_{k-1} < 0$, we have

$$a_{k-1} + \sum_{i=0}^{k-2} a_i x_{k-1-i} - \Theta < \sum_{i=0}^{k-2} a_i x_{k-1-i} - \Theta$$

which implies that $f(0, x_1, x_2, \ldots, x_{k-1}) = 1$ and $f(1, x_1, x_2, \ldots, x_{k-1}) = 0$. Since T is reversible, $f(0, x_1, x_2, \ldots, x_{k-1}) \neq f(1, x_1, x_2, \ldots, x_{k-1})$. Hence $A(\mathbf{a}, \Theta)$ is an antishift.

If $a_{k-1} = 0$, then T cannot be a bijection. Indeed for two different elements, we have

$$f(0, x_1, x_2, \ldots, x_{k-1}) = f(1, x_1, x_2, \ldots, x_{k-1})$$

thus T is not a bijection.

Proposition 4. *$A(\mathbf{a}, \Theta)$ is a shift if and only if*

$$[\sum_{i \leq k-2 \text{ and } a_i < 0} a_i - \Theta] < 0$$

and

$$[a_{k-1} + \sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta] \geq 0 \qquad (\alpha)$$

$A(\mathbf{a}, \Theta)$ is an antishift if and only if

$$[\sum_{i \leq k-2 \text{ and } a_i < 0} a_i - \Theta] \geq 0$$

and

$$[a_{k-1} + \sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta] < 0 \qquad (\beta)$$

Proof. Assume that $A(\mathbf{a}, \Theta)$ is a shift. For $\mathbf{x} = (x_1, x_2, \ldots, x_{k-1})$ in $\{0,1\}^{k-1}$ define $(0, \mathbf{x}) = (0, x_1, x_2, \ldots, x_{k-1})$ and $(1, \mathbf{x}) = (1, x_1, x_2, \ldots, x_{k-1})$. We have $f(0, \mathbf{x}) = 0$ and $f(1, \mathbf{x}) = 1$.

Define

$$\begin{aligned} \mathcal{P} &= \{i \in \{0, 1, 2, \ldots, k-2\} \text{ such that } a_i > 0\}, \\ \mathcal{N} &= \{i \in \{0, 1, 2, \ldots, k-2\} \text{ such that } a_i < 0\}. \end{aligned}$$

Let $\mathbf{y} = (y_1, y_2, \ldots, y_{k-1}) \in \{0,1\}^{k-1}$ be such that

$$y_i = \begin{cases} 0 & \text{if } k - i - 1 \in \mathcal{N} \\ 1 & \text{if } k - i - 1 \in \mathcal{P} \\ & \text{otherwise} \end{cases}$$

where $y_i = *$ means that y_i can take any value in $\{0,1\}$,

$$f(0, y_1, y_2, \ldots, y_{k-1}) = 1[\sum_{i \leq k-2 \text{ and } a_i < 0} a_i - \Theta] = 0$$

implies that

$$\sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta < 0$$

Let now $\mathbf{z} = (z_1, z_2, \ldots, z_{k-1})$ be such that $z_i = 1 - y_i$.

$$f(1, z_1, z_2, \ldots, z_{k-1}) = 1[a_{k-1} + \sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta] = 1$$

implies that

$$a_{k-1} + \sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta \geq 0$$

Conversely, let us assume that the formula (α) is verified.

Since $\sum_{i=0}^{k-2} a_i x_{k-1-i} - \Theta \leq \sum_{i \leq k-2 \text{ and } a_i > 0} a_i - \Theta < 0$, we deduce that $f(0, x_1, x_2, \ldots, x_{k-1}) = 0$ and hence $A(\mathbf{a}, \Theta)$ is a shift.

In a similar way we can show that $f(1, \mathbf{x}) = 1$. ∎

Theorem 3. *1. A reversible automaton $A(\mathbf{a}, \Theta)$ has only cycles of length L such that*

L divides k if $A(\mathbf{a}, \Theta)$ is a shift (i.e. $a_{k-1} > 0$),

L divides $2k$ if $A(\mathbf{a}, \Theta)$ is an antishift (i.e. $a_{k-1} < 0$),

2. The automaton defined by equation (1.1) cannot have a cycle of length 2^k.

Proof. (1) is an immediate consequence of the previous remarks. (2) If $A(\mathbf{a}, \Theta)$ has a cycle of length $L = 2^k$, then the associated operator T has also a cycle of length $L = 2^k$, and thus is bijective, which leads to a contradiction when k is greater than 1. ∎

Many properties of shift and antishift operators are given in [12]. Moreover, note that if we have an automaton with a geometric memory (i.e. case where $a_i = -(b^i)$ with $b > 0$) then for every $0 < b \leq 1$ the operator T is never bijective. But if we have $b \geq 2$ then there always exists $\Theta_{(k,b)} = \frac{b^k - 1}{2(b-1)}$ such that the operator T is bijective.

From proposition 3 we deduce that there exist only two different reversible memory systems: the shift and the antishift. Hence if a_{k-1} is positive the reversible system is equivalent to $A(\mathbf{a}, \Theta)$ with $\mathbf{a} = (0, 0, 0, \ldots, 0, 1)$ and $\Theta = 1/2$ (shift). Otherwise, if a_{k-1} is negative, the reversible system is equivalent to $A(\mathbf{a}, \Theta)$ with $\mathbf{a} = (0, 0, \ldots, 0, -1)$ and $\Theta = -1/2$ (antishift).

5. Pivot sums

We first study the case of a single pivot and then we shall treat the general case.

Definition 3. *i is 1-pivot if for every n, $x_{n+1} = 1$ implies $x_{n-i} = 1$.*

i is 0-pivot if for every n, $x_{n+1} = 0$ implies $x_{n-i} = 0$.

i is an antipivot if for every n, $x_{n-i} = 0$ is equivalent to $x_{n+1} = 1$.

Lemma 4. *i is 1-pivot if and only if*

$$\sum_{\substack{j=0 \\ j \neq i \text{ and } a_j > 0}}^{k-1} a_j - \Theta < 0$$

i is 0-pivot if and only if

$$a_i + \sum_{\substack{j=0 \\ j \neq i \text{ and } a_j < 0}}^{k-1} a_j - \Theta \geq 0$$

i is an antipivot if and only if

$$a_i + \sum_{\substack{j=0 \\ j \neq i \text{ and } a_j > 0}}^{k-1} a_j < \Theta \leq \sum_{\substack{j=0 \\ j \neq i \text{ and } a_j < 0}}^{k-1} a_j$$

Proof. Follows from direct computations. ∎

Theorem 4. *Let p be the period of a cycle of the automaton $A(\mathbf{a}, \Theta)$.*

If i is a 0-pivot or a 1-pivot then p divides $(i+1)$.

If i is an antipivot then p divides $2(i+1)$.

Proof. Assume that i is a 1-pivot. In order to prove that p divides $(i+1)$, we introduce the following operator

$$E_i(n) = x_{n+1} - x_{n-i}.$$

It is clear that E_i is a negative function; i.e., for every n, $E_i(n) \leq 0$. Call q the length of the transient of the trajectory and assume that there exists an integer $n > q + i$ such that $E_i(n) < 0$. We have then $x_{n+1} < x_{n-i}$. Since E_i is a negative function, we have

$$x_{n-i+p(i+1)} \leq x_{n-i+(p-1)(i+1)} \leq \ldots \leq x_{n+1} < x_{n-i}.$$

Since p is the period of the trajectory, we have $x_{n-i+p(i+1)} = x_{n-i}$. Thus we get a contradiction. We deduce that $E_i(n) = 0\ \forall n \geq i+q$ which implies that the period divides $(i+1)$.

The proof for the 1-pivot and the antipivot is similar. ■

We shall now treat two cases of pivot sum:

Definition 4. *Let $\mathcal{T}$ be a subset of $\{0, 1, \ldots, k-1\}$.*

1. *$\mathcal{T}$ represents 1-pivot sum if*

$$x_{n+1} = 1 \Rightarrow \forall i \in \mathcal{T}\ x_{n-i} = 1.$$

2. *$\mathcal{T}$ represents 0-pivot sum if*

$$x_{n+1} = 0 \Rightarrow \forall i \in \mathcal{T}\ x_{n-i} = 0.$$

Lemma 5. 1. *$\mathcal{T}$ represents 0-pivot sum if and only if*

$$\sup_{i\in\mathcal{T}}\Big[\sum_{\substack{j=0 \\ j\neq i \text{ and } a_j>0}}^{k-1} a_j - \Theta\Big] < 0$$

2. *$\mathcal{T}$ represents 1-pivot sum if and only if*

$$\inf_{i\in\mathcal{T}}\Big[a_i + \sum_{\substack{j=0 \\ j\neq i \text{ and } a_j<0}}^{k-1} a_j - \Theta\Big] \geq 0$$

Proof. Follows from a direct computation. ■

Corollary 4. *If we have 0-pivot sum or 1-pivot sum, then the period p always divides $\gcd_{i\in\mathcal{T}}(i+1)$ (gcd = great common divisor).*

Proof. Follows directly from the fact that each $i \in \mathcal{T}$ constitutes a pivot of the system. ■

Corollary 5. *Let $\mathcal{T}$ represent t-pivot sum (with $t = 0$ or $t = 1$).*

If $\mathcal{T}$ contains two consecutive integers, then the system has only fixed points.

Proof. Obvious. ■

6. Conjectures

If the coupling coefficients a_i are positive, then the period of each cycle of a given automaton $A(\mathbf{a}, \Theta)$ is less than or equal to k.

In the general case, the period of each cycle of a given automaton $A(\mathbf{a}, \Theta)$ is less than or equal to $2k$.

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