

Simulations of Mixtures of Two Boolean Cellular Automata Rules

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Abstract. We consider cellular automata on a square lattice with nearest-neighbor inputs. Our automata are quenched mixtures of two Boolean functions. We investigate if there is a phase transition as a function of the degree of mixing, particularly if one chooses one rule as forcing and the other as non-forcing.

1. Introduction

The Kauffman model [1,2,3] is a cellular automata [4] defined on a lattice where one associates a binary variable ρ_i to each site which is either zero or unity. The time evolution for each site is determined by a rule randomly picked among the complete set of Boolean functions of K inputs. In the present work, the K inputs are the nearest neighbors on a square lattice and in one dimension.

This model presents under certain conditions a phase transition between a frozen phase and a chaotic one. In the frozen phase, a disturbance cannot propagate, whereas in the chaotic phase it does.

Derrida and Stauffer [5] have explored the Kauffman case, where they used instead of K as varying parameter the probability p for a Boolean function to have the value 1 (most early studies were made for $p = 0.5$). They treat the annealed case (in each iteration, the Boolean functions are changed) and the quenched one (once the Boolean functions are chosen for $t = 0$ they are kept for all times). In both cases, a phase transition was found.

A Boolean function is said forcing [1–3] if at least one input site assuming a determined value, determines the output of the function, for example the logical OR for the value 1 and AND for the value 0.

We know that among the Boolean functions in the Kauffman model there are forcing functions and non-forcing ones. Neighboring forcing functions

tend to correlate the system favoring an ordered phase whereas non-forcing structures tend to scatter 0's and 1's, disorganizing the system. So there exists competition between rules in Kauffman's model.

A damage is defined [6] as the number of sites differing as a result of the time-evolution of a single error introduced in the system, if initially one ρ_i is changed. Of course, damage (in the sense of genetic mutation) propagates easier in weakly correlated systems. It has been proposed [1-3,7] that competition between forcing and non-forcing functions is responsible for the phase transition in cellular automata like the Kauffman model. In this work, we want to investigate this question in more detail.

We consider the quenched case and mix [7] *only two* rules $F1$ and $F2$, for example one forcing and another non-forcing. Specifically, we consider only symmetric rules, i.e., $F(1, 2, 3, \dots, k-1, k) = F(1, k, k-1, \dots, 3, 2)$ where 1 is the central site and $2, 3, \dots, k-1, k$ are its neighbors. We take as varying parameter the probability p to have rule $F1$ on a given site and $1-p$ to have $F2$.

2. Method

A way to characterize the chaotic phase is through the time-development of the normalized Hamming distance $\Psi(t)$ between two configurations $\{\rho_i(t)\}$ and $\{\varrho_i(t)\}$ on which we apply simultaneously the same set of functions $\{F_i\}$. The distance $\Psi(t)$ is defined by

$$\Psi(t) \equiv \sum_{i=1}^N (\rho_i(t) - \varrho_i(t))^2 / N$$

where N is the number of sites of the lattice. Initially, $\rho_i = \varrho_i$ except for one randomly selected site.

We thus start with two configurations having for $t = 0$ a distance $1/N$ between each other and calculate $\Psi_\infty \equiv \lim_{t \rightarrow \infty} \Psi(t)$. Since the frozen phase is insensitive to an initial disturbance, we have in this case $\Psi_\infty = 0$ for the limit $\Psi(t = 0) \rightarrow 0$. In the chaotic phase, the limit distance Ψ_∞ is different from zero. So one can use the distance Ψ_∞ as a disorder parameter.

We calculate the distance Ψ_∞ if the initial disturbance tends to zero [5]. Since one works with finite systems, one has to perform this via an extrapolation. Following the lines of Stanley et. al. [6] we take three different initial configurations, namely A, B, and C, constructed as follows.

Configuration A The original "undamaged" configuration.

Configuration B Differs from configuration A in one randomly chosen site.

Configuration C Differs from configuration B in another randomly chosen site and from configuration A in these two sites.

Then we extrapolate to zero initial damage by the equation

$$\Psi_{\infty} = \Psi_{\infty}(A, B) + \Psi_{\infty}(B, C) - \Psi_{\infty}(A, C).$$

We calculate Ψ_{∞} for various mixtures of rules, using periodic boundary conditions, by numerical simulation using Multi-Spin-Coding techniques [8], implemented on a Cray XMP and getting a speed of $7 * 10^7$ updates of the three configurations per second. For improvements in the method see [12].

The initial configurations are constructed randomly having a certain concentration q of 1's. In our data, we used $q = 0.5$ and verified in some cases that the results are unaffected for other values of q except for $q = 0$ and $q = 1$.

In order to calculate Ψ_{∞} one must let the system evolve to an equilibrium. We monitor $\Psi(t)$ as function of time t and see it saturating towards Ψ_{∞} after a characteristic time τ_0 which is of order $\tau_0 = 500 - 10000$ for the systems that we considered. Thus, we iterated τ times where τ was chosen to be several times τ_0 .

Two averages must be performed: one over different initial configurations and one over different distributions of rules. We perform both averages at once by choosing for each sample as well a new set of rules as a new initial configuration; typically, we average over $M = 20 - 500$ samples.

Finally, in order to take the thermodynamic limit, we simulate systems of different linear sizes L . To optimize vectorization, our sizes must be multiples of 64 and the smallest choice is $L = 192$; our largest L was 40000 in one and 768 in two dimensions.

3. Results

We start treating the one-dimensional problem. We choose as Boolean functions $F1$, the generalized OR (which is true (1) if at least one of its K arguments is true) and as $F2$ the generalized XOR (which is true if an odd number of its arguments is true). We study the cases $K = 3$ (nearest neighbors and central site) and $K = 5$ (nearest and next-nearest neighbors and central site) and verify that there is no phase transition. In figure 1a, we show the size-dependence of the order parameter Ψ_{∞} as a function of p and L . We see that for $p = 0$, Ψ_{∞} goes to zero as the size L goes to infinity. This shows that the rule XOR, although being non-forcing, is frozen; i.e., $\Psi_{\infty}(p = 0) = 0$ in the thermodynamic limit. Another interesting point is the fact that there exists a maximum in the curve of figure 1a. The height of this maximum goes to zero with increasing L . In summary, this one-dimensional system does not spread its damage over infinite distances and thus is not chaotic.

Similarly, we studied the mixture of OR and XOR on the square lattice. Again, pure XOR is frozen as is OR. For the mixture of OR and XOR, however, for $0 < p < 0.4$ there is a chaotic phase as seen in figure 1b. The points do not show a significant size dependence, but there are strong statistical fluctuations: for some initial configurations $\Psi_{\infty} = 0$ and for others $\Psi_{\infty} \neq 0$. This is what one would expect: if the two initially damaged [9] sites and their neighbors happen to be not susceptible to damage, the initial

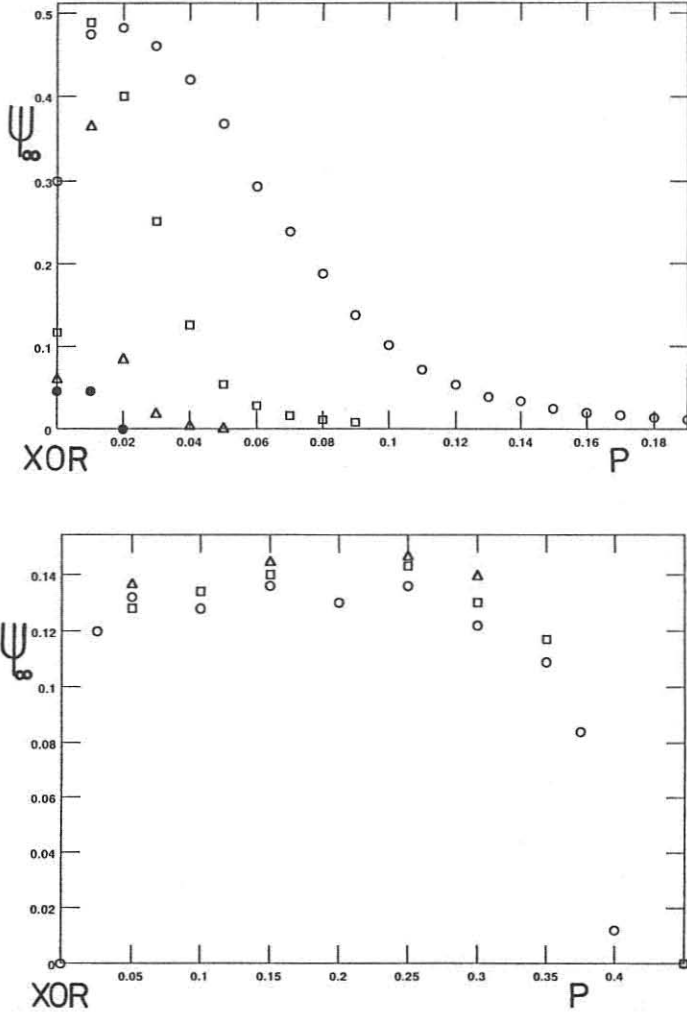


Figure 1: (a) Disorder parameter Ψ_∞ for a one-dimensional system in which rules XOR and OR are mixed with $K = 5$ as a function of p for different sizes L and number of runs M , after $\tau = 10000$ time steps: $L = 40000, M = 20$ (full circles); $L = 8000, M = 100$ (triangles); $L = 1600, M = 200$ (squares); and $L = 320, M = 500$ (open circles). (b) Ψ_∞ for a mixture of XOR and OR on a square lattice, as a function of p , $\tau = 10000$: $L = 192, M = 400$ (circles), $L = 256, M = 200$ (squares), $L = 384, M = 200$ (triangles). Ψ_∞ is zero in the nonchaotic phase and nonzero in the chaotic phase, for the limit $L \rightarrow \infty, \tau \rightarrow \infty$. Thus figures 1 through 3 show the transition to chaos.

damage can die out quickly. The average over many initial configurations however seems to yield a value of $\Psi_\infty \sim 0.13$.

Next, we perform a more systematic study of other rules on the square lattice. We take as non-forcing rule the function Q2R [8,10], where a site is damaged if and only if the two of its neighbors are true (1) and the two others are false (0) and is given by

$$F2 = \text{XOR}[f(\rho_1, \rho_2, \rho_3, \rho_4), \rho(i, j)]$$

where $\rho(i, j)$ is the central site, $\rho_1, \rho_2, \rho_3, \rho_4$ are its four neighbors, and

$$f = \text{OR}(f_a, f_b).$$

Here, f_a and f_b are defined as follows:

$$f_a = \text{AND}[\text{XOR}(\rho_1, \rho_2), \text{XOR}(\rho_3, \rho_4)]$$

and

$$f_b = \text{AND}[\text{XOR}(\rho_1, \rho_3), \text{XOR}(\rho_2, \rho_4)]$$

As presented in reference [6], the Q2R rule leads to chaotic behavior, at least for the range of parameters investigated there.

We analyze the phase diagram depending on the probability p to choose a given function $F1$, $1 - p$ being the probability for the function $F2$ (here, Q2R).

We present the results for the case

$$F1 = \text{OR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)]$$

We can see in figure 2 that in the thermodynamic limit for $p > 0.02$ one certainly is in the frozen phase. For $p > 0.01$, the value of Ψ_∞ decreases for increasing L and presumably one is in the frozen phase, too, for $L \rightarrow \infty$. For $p < 0.01$, the situation is less clear; either there is a chaotic phase with a $\Psi_\infty \sim 0.13$ or there is an extremely slow critical size dependence and Ψ_∞ ultimately goes to zero in the thermodynamic limit. For $p = 0$, Ψ_∞ equals 0.5. In conclusion, we cannot exclude a very small chaotic phase close to $p = 0$.

We present below twelve other rules $F1$ combined with Q2R where we have qualitatively verified similar results as in figure 2: for nearly all values of p there is clearly a frozen phase and only very close to $p = 0$ there is a region of slow convergence in L and τ_0 , which might be a crossover region.

$$F1 = \text{OR}[\text{AND}(\rho_1, \rho_2, \rho_3, \rho_4), \rho(i, j)]$$

$$F1 = f$$

$$F1 = \text{OR}[f, \rho(i, j)]$$

$$F1 = \text{XOR}[f_a, f_b, \rho(i, j)]$$

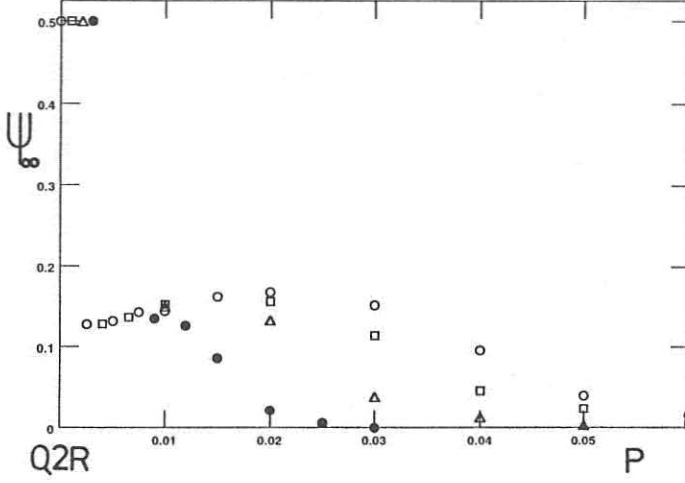


Figure 2: Ψ_∞ for a mixture Q2R and OR on a square lattice, as a function of p , $M = 200$ and $10000 < \tau < 40000$. $L = 768$ (full circles); $L = 384$ (triangles); $L = 256$ (squares); and $L = 192$ (open circles). At $p = 0$ all points are at $\Psi_\infty = 1/2$.

$$F1 = f_b$$

$$F1 = \text{XOR}[f_b, \rho(i, j)]$$

$$F1 = \text{AND}[f_a, \text{NOT}(f_b)]$$

$$F1 = I \text{ (identity)}$$

$$F1 = \text{OR}\{\text{XOR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)], \text{OR}[\text{AND}(\rho_1, \rho_2), \text{AND}(\rho_3, \rho_4)]\}$$

$$F1 = \text{OR}\{\text{XOR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)], \text{OR}[\text{AND}[\rho_1, \rho_2, \rho(i, j)], \text{AND}[\rho_3, \rho_4, \rho(i, j)]]\}$$

$$F1 = \text{OR}\{\text{XOR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)], \text{AND}[\text{OR}(\rho_1, \rho_2, \rho_3, \rho_4), \rho(i, j)]\}$$

$$F1 = \text{XOR}\{\text{AND}[f_a, \text{NOT}(f_b)], \rho(i, j)\}$$

For all these functions, we verified the existence of the frozen phase everywhere except in a very close vicinity of $p = 0$.

In contrast to the former case, we now present another class of functions $F1$ combined with Q2R, where we have along the whole interval of p except for $p = 1$ the chaotic phase

$$F1 = \text{XOR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)]$$

$$F1 = \text{OR}\{\text{XOR}[\rho_1, \rho_2, \rho_3, \rho_4, \rho(i, j)], \text{AND}[\rho_1, \rho_2, \rho_3, \rho_4]\}$$

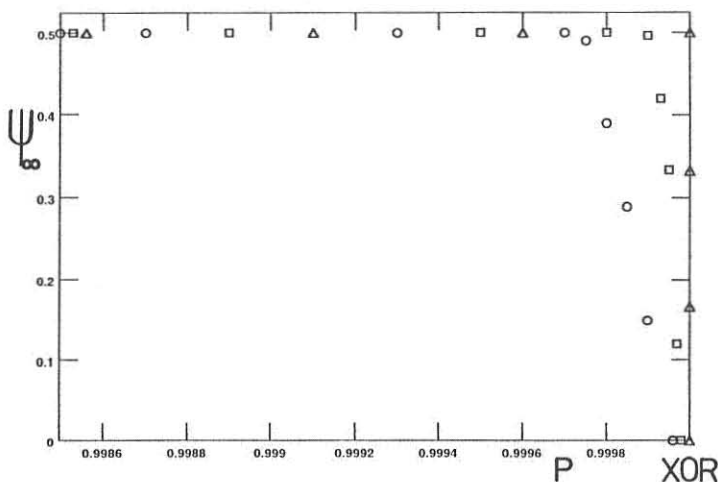


Figure 3: Ψ_∞ for a mixture of Q2R and XOR on a square lattice, as a function of p , $\tau = 10000$ and $M = 10$. $L = 192$ (circles), $L = 320$ (squares), the triangles give the expected limit for infinite system size. Note the pathological behavior of pure XOR: $\Psi_\infty = 0$ for $p = 1$ for all L .

In figure 3, we show the results corresponding to the first of these three functions. We have $\Psi_\infty(p < 1) = 0.5$ and $\Psi_\infty(p = 1) = 0$ for $L \rightarrow \infty$, which indicates the absence of the transition at $p < 1$. Note the small width of the crossover region.

To test our program, we also rewrote it without Multi-Spin-Coding and applied it to the mixture of rules studied in reference [11]. We confirmed the result there, and in addition found a phase transition near $p = 0.45$, if we combined 1-XOR (probability p) with the complement of the other rule of [11] (with and without Multi-Spin-Coding). Such transitions are useful candidates to study the fractal dimensions of damage spreading [6,9]. Moreover, in the case of a pure XOR or pure 1-XOR rule, we found a completely regular and deterministic spread of damage, with fourfold symmetry, even with a random initial distribution of spins. This pathological behavior may explain some of the difficulties described above for the limits $p \rightarrow 0$ or $p \rightarrow 1$.

4. Conclusions

We studied the relevance that forcing and non-forcing rules can have for the appearance of a phase transition.

We mixed only two symmetric rules $F1$, $F2$ chosen with probabilities p and $1 - p$ respectively instead of mixing all rules as in the Kauffman model.

Usually, the rule $F1$ was a rule that leads the system to a frozen phase whereas $F2$ leads to a chaotic one.

We found the following results:

Not all non-forcing rules lead to a chaotic phase as is the case for XOR in the one-dimensional or on the square lattice.

There is a transition between a chaotic and nonchaotic phase if XOR is diluted with OR on the square lattice; this phase shows, however, a strong dependence on the choice of the initial configuration.

Mixing the chaotic rule Q2R with frozen rules one does not seem to give a phase transition at an intermediate value of p but only at the extremes $p \rightarrow 0$ or $p \rightarrow 1$. What exactly happens in these extremes is not easy to discern because of long relaxation times, but in some cases as the one of figure 2, as complete scenario is suggested. The opposite examples of [7] and [11] thus may be the exception rather than the rule.

We conclude that mixing only two rules is not enough to explain the phase transition in Kauffman's model or other inhomogeneous cellular automata [6]. XOR is less strongly frozen than OR. So XOR mixed with Q2R gives a chaotic phase for all p except $p = 1$ while OR mixed with Q2R gives a frozen phase except very close to the pure Q2R limit. To obtain a phase transition at an intermediate value of p , all these different rules, frozen to a different extent, must be mixed. In addition, we only considered symmetric rules. Asymmetric rules might also be important to understand the phase transition in the Kauffman model and elsewhere.

In summary, in contrast to the original hopes [1,2,7] and to recent results for the "linear" Kauffman model [13], the transition to chaos is not reliably described by forcing functions.

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