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On σ-Automata

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Abstract. A σ-automaton is a particularly simple type of cellular automaton on a graph: each cell is either in state 0 or 1 and the next state of a cell is determined by adding the states of its neighbors modulo two. Using algebraic and graph-theoretic techniques, questions such as reversibility and existence of predecessor configurations of these automata will be studied. We derive some general results for product graphs. In particular, we give a simple characterization of the number of predecessors of a configuration in rectangular grids, cylinders, and tori.

1. Introduction and definitions

A cellular automaton in its most general form is a discrete dynamical system. Its components, called cells, are interconnected in a fixed way and can assume finitely many possible states. A configuration of the system is an assignment of states to all cells. Every configuration determines a next configuration via a transition rule that is local in the following sense: the state of a cell at time $t+1$ depends only on the states of its neighbors at time t. In this paper, we will consider cellular automata with an extremely simple transition rule: there are only two possible states 0 and 1, and the state of cell at time $t+1$ is defined to be the sum of the states of its neighbors at time t, calculated modulo two. A cell may or may not be included in its own neighborhood. The next configuration is determined by applying this rule simultaneously to all cells. On the other hand, we will allow arbitrary adjacencies between the cells of the automaton. Cellular automata of this type will be called *σ-automata*. A typical example of a σ-automaton is a traditional one-dimensional cellular automaton with rule 90 or rule 150 (see [3,8]): there are infinitely many cells v_i, $i \in Z$, and the neighborhood of cell v_i consists of v_{i-1} and v_{i+1} for rule 90, v_{i-1}, v_i, and v_{i+1} for rule 150. The evolution of a configuration that has exactly one cell in state 1 and all the others in state 0 leads to well-known fractal patterns; see e.g. [8]. σ-automata on trees were first considered by Lindenmayer in [12] and occur also in [1,4].

We do not wish to restrict ourselves to the simple neighborhood situations of one-dimensional cellular automaton. Therefore, we will use graphs to describe the underlying mesh of cells and their connections in a σ-automaton. We begin with a somewhat less informal definition. Our graph theoretic terminology follows Berge (see [2]). For our purposes, a *graph* is best defined as a pair $G = \langle V, E \rangle$ where $E \subseteq V \times V$. These structures allow for self loops and are sometimes referred to as pseudo graphs; they correspond to 1-graphs in [2]. Furthermore, we will insist that G is locally finite, i.e. every vertex in G is adjacent to only finite many vertices. It is convenient to identify E with the adjacency matrix of G construed as a matrix in $\prod_{V \times V} F_2$. Here $F_2 = \{0, 1\}$ is the two-element field and there is a 1 in the u^{th} row and v^{th} column of E iff there is a directed edge from vertex u to vertex v in G.

A vertex u is a predecessor of v iff there exists an edge (u, v) in G. The collection of all predecessors of v will be denoted by

$$\Gamma_G(v) := \{u \in V | (u, v) \in E\}.$$

Note that $\Gamma_G(v)$ may or may not include v depending on whether v has a self loop in G or not. We will refer to $\Gamma_G(v)$ as the *neighborhood* of v in G. A *configuration* of G is a function

$$X : V \to F_2.$$

The collection of all configurations of G will be denoted by C_G. Define the *transition rule* $\sigma_G : C_G \to C_G$ by

$$\sigma(X)(v) := \sum_{u \in \Gamma(v)} X(u).$$

$A = \langle G, \sigma_G \rangle$ is called the *σ-automaton* on G. A σ-automaton $A = \langle G, \sigma_G \rangle$ is *symmetric* iff the adjacency matrix E of G is symmetric, thus symmetric σ-automata arise from undirected graphs. Let G be an undirected graph without self loops and D a subset of V. Define $G(D)$ to be the graph obtained from G by adding self loops at all vertices in D. σ-automata of the form $\langle G, \sigma_{G(V)} \rangle$ or $\langle G, \sigma_{G(\phi)} \rangle$ are called *Lindenmayer automata* on G.

To lighten notation we will usually omit the subscript G and write $\Gamma(v)$, σ, $\langle G, \sigma \rangle$ and so forth. Also we will write σ^+ for $\sigma_{G(V)}$ and σ^- for $\sigma_{G(\phi)}$. We frequently identify singletons $\{v\}$ with v, $v \in V$. Thus, $X := u + v$ defines the configuration $\{u, v\}$ if $u \neq v$ and the empty set otherwise. We will write 0 for the empty set and 1 for V as members of C_G, so $0(v) = 0$ and $1(v) = 1$ for all v in V. Let us agree on some notation for graphs: $P_m(C_m)$ will denote the undirected path graph (cycle) on m points. Their vertex sets are assumed to be $\{1, 2, \ldots, m\}$. Lastly, for an integer n let $[n] := \{1, 2, \ldots, n\}$.

Configurations are conveniently identified with subsets of V, i.e. X is identified with $\{v \in V | X(v) = 1\}$. Observe that algebraically C_G is a vector space over F_2, $C_G = \prod_V F_2$; addition here amounts to taking symmetric differences. We will call this space the *configuration space*. Furthermore, σ is a linear map from the configuration space to itself (such rules are called additive in [3]). If one thinks of configuration X as a column vector over F_2, it is obvious from the definitions that

$$\sigma(X) = E^T \cdot X$$

where E^T is the transpose of the adjacency matrix of G. The arithmetic is understood to be over the field F_2. Also note that σ is an example of a uniform rule: σ is defined for all graphs regardless of the particular types of neighborhoods that occur. Other rules of this type are studied in [5].

From the point of view of dynamical systems, one of the basic questions about a σ-automaton $A = \langle G, \sigma \rangle$ is whether rule σ is *reversible* on G: can configuration X be reconstructed from $\sigma(X)$ or, in other words, is the map $\sigma : C_G \to C_G$ injective. As σ is linear, this is equivalent to the question whether σ has trivial kernel. In terms of the transition diagram, reversibility is equivalent with every node having in-degree at most 1. Note that rule σ is locally irreversible in the sense that different configurations can lead to the same state in one particular cell in the next generation. However, globally this rule may well be reversible. A general characterization of those graphs G for which σ_G is injective seems rather difficult. Some steps in this direction for Lindenmayer automata were taken in [1] and [4]. In this paper, we will focus on product graphs like grids $P_m \times P_n$, cylinders $C_m \times P_n$, and tori $C_m \times C_n$ (see section 4 for definitions). For all these graphs, reversibility with respect to rule σ^- depends on simple number theoretic properties of m and n. For example, we will show that a $m \times n$ grid is reversible iff $m + 1$ and $n + 1$ are relatively prime. A configuration X is a *predecessor* of Y iff $\sigma(X) = Y$. We will show that the number of predecessors of a configuration in a $m \times n$ grid is (either 0 or)

$$2^{gcd(m+1,n+1)-1}.$$

Hence, the number of predecessors depends not so much on the size of the grid but rather on number-theoretic properties of m and n. Note that one can determine the reversibility of a $m \times n$ grid in $O((\log_2 mn)^3)$ steps using the Euclidean algorithm. By comparison, the brute force approach based on computing the determinant of the adjacency matrix of the graph is polynomial in n and m. Similarly, a $m \times n$ cylinder is reversible iff m and $n + 1$ are relatively prime and either both m and $n + 1$ are odd or the exponent of 2 in the prime decomposition of m is strictly larger than the exponent of 2 in the prime decomposition of $n + 1$. Our proof hinges on the fact that rule σ^- displays simple periodicity properties on all these graphs. Unfortunately, the behavior of rule σ^+ is much more complicated; no analogous analysis for rule σ^+ is available at this point.

Another basic problem is to determine which configurations have predecessors under rule σ. A σ-automaton is *complete* iff every configuration has a predecessor. In terms of the transition diagram, this means that every configuration has indegree at least 1. Note that by the linearity of σ predecessor existence in a finite σ-automaton is closely related to reversibility of the automaton: rule σ is reversible on G (i.e., rule σ_G is injective) iff $\langle G, \sigma \rangle$ is complete (i.e., rule σ_G is surjective). More generally, one would like to characterize configurations which have predecessors in the t^{th} generation. For

Lindenmayer automata, the configurations with predecessors can be characterized in terms of the kernel of σ_G. For certain simple graphs, this allows to determine the configurations that possess predecessors completely.

The remainder of this paper is organized as follows. In section 2, we will exploit the linearity of rule σ to establish some general results about the transition diagram of a σ-automaton. We briefly indicate how to lift results from finite to infinite σ-automata. In section 3, the focus is on finite Lindenmayer automata on product graphs, typically grids and cylinders. We introduce a family of linear operators that allow to determine the reversibility of general product graphs. In particular for grids, cylinders, and tori with rule σ^- a complete analysis is given. We provide some numerical data that point towards the difficulties of a similar approach for rule σ^+.

2. The transition diagram of a Lindenmayer automaton

The action of rule σ_G on the configuration space C_G is best expressed graphically by means of the *transition diagram* $\mathcal{C}_G$ of G. Formally, the transition diagram is a directed graph that has as points the configurations in G and an arc from X to Y iff X is a predecessor of Y, $\sigma(X) = Y$. As σ is linear, the transition diagram is highly uniform. For example, suppose configuration X has predecessor Y. Then the collection of all predecessors of X is the affine subspace $\sigma^{-1}(X) = Y + \ker(\sigma)$; thus the indegree of any configuration X in $\mathcal{C}_G$ is either 0 or 2^d, $d := \dim(\ker(\sigma))$. The out-degree is of course 1, so the connected components of $\mathcal{C}_G$ are all unicyclic (i.e., they contain at most one cycle). Tracing a path in $\mathcal{C}_G$ shows the evolution of one particular configuration. Clearly, automaton $A = \langle G, \sigma \rangle$ is reversible iff the connected components of $\mathcal{C}_G$ are cycles or infinite paths. For the special case where $G = C_N$ is a cycle on N points $\mathcal{C}_G$ was studied extensively in [3].

Define the set of all t^{th} generation predecessors of X by $\sigma^{-t}(X) := \{Y | \sigma^t(Y) = X\}$. Thus, $\sigma^{-t}(X)$ is the collection of all vertices in $\mathcal{C}_G$ that have a path of length t to X. Note that $\sigma^{-t}(X)$ is an affine subspace of C_G; in particular, if X has a t^{th} generation predecessor Y then $\sigma^{-t}(X) = Y + \ker(\sigma^t)$. Define the co-orbit of X to be the collection of all predecessors of X, $\text{co-orb}(X) := \bigcup_{t \geq 0} \sigma^{-t}(X)$. Similarly, the orbit of X is defined as $\text{orb}(X) := \{\sigma^t(X) | t \geq 0\}$. A configuration X is *persistent* iff for all $t \geq 0$, $\sigma^{-t}(X) \neq \phi$, and semi-persistent iff there exists a persistent configuration Y in the orbit of X. Thus, a persistent configuration must lie on a co-infinite path in $\mathcal{C}_G$ or on a cycle. Fix such a path or cycle, i.e., a sequence $(X^{-t} : t \geq 0)$ of configurations such that $\sigma(X^{-t-1}) = X^{-t}$ for all $t \geq 0$ and $X^0 = X$. Furthermore, let us agree that whenever X lies on a cycle in $\mathcal{C}_G$, the configurations X^{-t} are also chosen on that cycle. (It may happen that X lies on a co-infinite path and on a cycle.) Note that in a finite configuration space, every configuration is semi-persistent and all persistent configurations lie on cycles.

For all semi-persistent configurations Y define the height of Y by

$$h(Y) := \min(t \geq 0 | \sigma^t(Y) \text{ is persistent}).$$

Also, let $h(G) := \max(h(Y)|Y \text{ in } C_G \text{ semi-persistent})$ and define for persistent configuration X

$$T(X) := \{Y \in \text{co-orb}(X)|\sigma^{h(Y)}(Y) = X\}.$$

Note that $T(X)$ is a tree with root X; the subtress generated by the $2^d - 1$ sons of X are full 2^d-ary trees (every node has either 2^d sons or none at all). A connected component of $\mathcal{C}_G$ thus consists of a cycle and trees anchored on that cycle.

The following is a generalization of lemma 3.3 in [3].

Lemma 2.1. *Let $A = \langle G, \sigma \rangle$ be a σ-automaton. For any two persistent configurations X and Y, the trees $T(X)$ and $T(Y)$ in the transition diagram $\mathcal{C}_G$ are isomorphic.*

Proof. For all configurations Z in $T(X)$, define

$$f(Z) := Z + X^{-h(Z)} + Y^{-h(Z)}.$$

Observe that $f(\sigma(Z)) = \sigma(f(Z))$ for Z, $\sigma(Z) \in T(X)$. Also, $\sigma^{h(Z)}(f(Z)) = \sigma^{h(Z)}(Z) + \sigma^{h(Z)}(X^{-h(Z)}) + \sigma^{h(Z)}(Y^{-h(Z)}) = X + X + Y = Y$.

Thus, $f(Z)$ is in the co-orbit of Y and $s := h(f(Z)) \leq h(Z) := t$. Suppose for the sake of a contradiction that $s < t$. Then for some $r \geq 0$

$$Y^{-r} = \sigma^s(f(Z)) = \sigma^s(Z) + X^{s-t} + Y^{s-t}.$$

But then $\sigma^{t-s}(Y^{-r}) = \sigma^t(Z) + X + Y = Y$. This implies that either $t - s = r$ or Y lies on a cycle and $t - s$ divides r. In either case, $Y^{-r} = Y^{s-t}$ which finally yields $\sigma^s(Z) = X^{s-t}$. Thus, $h(Z) \leq s$ and we have a contradiction. Furthermore, we have $f(Z) \in T(Y)$.

Now let f' be defined as f but with domain $T(Y)$ and range a subset of $T(X)$. As $h(f(Z)) = h(Z)$ and $h(f'(Z)) = h(Z)$, we have $f(f'(Z)) = Z$ and $f'(f(Z)) = Z$. Thus, f and f' are both bijections and we are through. ∎

The last lemma clearly holds in any vector space over F_2 with some linear operator σ. To address the specific properties of $\mathcal{C}_G$ one can frequently use the automorphisms of the underlying graph G. Suppose $F : V \to V$ is an automorphism of G. F acts naturally on the configuration space C_G by setting $F(X) := \sum_{x \in X} F(x)$ for x in C_G. Observe that F commutes with $\sigma : F(\sigma(X)) = \sigma(F(X))$ for any configuration X. The automorphisms of G are said to *act transitively* on a subset C_0 of C_G iff $\forall X, Y \in C_0 \exists F$ automorphism $(F(X) = Y)$. For example, let G be the cycle on N points and $C_0 := \{v|v \in V\}$. A cyclic shift is an automorphism of G, thus the automorphism group of G acts transitively on C_0. An automorphism F of G is called an *involution* iff $F \circ F = \text{id}$. The importance of involutions comes from the following simple observation.

Proposition 2.2. *Let F be an involution of G. If the kernel of σ on G is non-trivial, then there exists a non-trivial configuration X in the kernel of σ such that X is invariant under F.*

Proof. Suppose $X \neq 0$ is in the kernel of σ. If X is invariant under F we are done, so suppose $X \neq F(X)$. Let $Y := X + F(X) \neq 0$. Then $\sigma(Y) = \sigma(X) + \sigma(F(X)) = 0 + F(\sigma(X)) = 0$ and $F(Y) = F(X) + F^2(X) = F(X) + X = Y$, as desired. ∎

Lemma 2.3. *Let $A = \langle G, \sigma \rangle$ be a σ-automaton. Suppose the automorphisms of G act transitively on a basis of the kernel of σ. Then $T(0)$, the co-orbit of 0, is a tree consisting of $2^d - 1$ subtrees which have as roots the non-trivial predecessors of 0. All these subtrees are complete 2^d-ary trees of the same depth.*

Proof. It follows from our transivity assumption that the subtrees rooted at X_i are all isomorphic. Let us write $\#_r := |\sigma^{-r}(0)|$, $r \geq 1$. Note that $\#_r \leq (2^d - 1)2^{d(r-1)}$ by counting. Thus, it suffices to show that equality holds. This is trivial for $r = 1$. Proceeding by induction, we may assume equality holds for all $r' < r$.

Now suppose $\sigma^r(Z) = X_i \in K_G$. Then, $\sigma^{-r}(X_i)$ is the affine space $Z + \sigma^{-r}(0)$. Hence $\#_r = (2^d - 1) \cdot (1 + \sum_{r'<r} \#_{r'}) = (2^d - 1)\ (1 + \sum_{r'<r} (2^d - 1)\ 2^{d(r'-1)}) = (2^d - 1)\ 2^{d(r-1)}$ are we are done. ∎

A typical example for a σ-automaton satisfying the hypothesis of the last lemma is again the cycle on N points C_N. The kernel of σ^+ on C_N for $N \equiv 0$ (mod3) is generated by $X_1 = 1 + 2 + 4 + 5 + \ldots + (N-2) + (N-1)$ and $X_2 := 2 + 3 + 5 + 6 + \ldots + (N-1) + N$. Let S be the cyclic shift operator on C_N. Then $S(X_1) = X_2$ and $S^2(X_2) = X_1$. Also note that the hypothesis is trivially satisfied whenever the dimension of the kernel is 1.

Example 1

Consider the graph $G = \langle N, E \rangle$ where $E := \{(v, v+1) | v \geq 0\}$. Rule σ here amounts to a left-shift. Clearly, every configuration is persistent. Also note that the kernel of σ has basis $\{0\}$; thus the hypothesis of the last proposition is trivially satisfied. It follows that $T(X) - \{X\}$ is an infinite complete binary tree for all configurations X. Furthermore, there are exactly two cycles of length 1 in $\mathcal{C}_G$: they are generated by the two fixed points of σ on G, namely N and ϕ. $T(\phi)$ is the class of all finite subsets of N, and $T(N)$ is the class of all cofinite subsets of N.

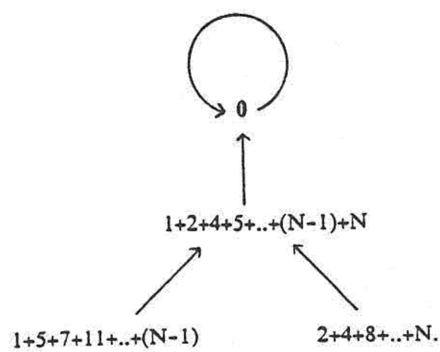
We now turn to symmetric automata. One can define an "inner product" on C_G by setting $\langle X, Y \rangle := |X \cap Y| \bmod 2$. Let X and Y be two configurations; X is called *perpendicular* to Y iff $\langle X, Y \rangle = 0$. The following theorem characterizes configurations that have t^{th} generation predecessors for finite symmetric σ-automata; a proof can be found in [4].

Theorem 2.4. *Let $A = \langle G, \sigma \rangle$ be a finite symmetric σ-automaton, $t \geq 0$. Then a configuration X occurs in the t^{th} generation, i.e., $X = \sigma_G^t(Y)$ for some configuration Y, iff X is perpendicular to the kernel of σ^t.*

Corollary 2.5. *Let $A = \langle G, \sigma \rangle$ be a finite symmetric σ-automaton. Then a configuration X is persistent iff X is perpendicular to the co-orbit of 0.*

Example 2

Consider the Lindenmayer automaton on P_N, the path on N points, with rule σ^+. for $N \not\equiv 2 \pmod 3$ the kernel of σ^+ on P_N is trivial; see lemma 4.3 of [4]. For $N \equiv 2$ (mod3) the kernel of σ^+ is generated by $X_1 = \sum_{x \not\equiv 0 \pmod 3} x$. Thus for these N exactly the configurations of the form $Z = Z_0 \cup Z_1$ with $Z_1 \subseteq X_1$, $|Z_1|$ even, and $Z_0 \subseteq 3+6+\ldots+(N-2)$ arbitrary have a predecessor under rule σ^+. Thus, exactly half the configurations have a predecessor in $\langle P_N, \sigma^+ \rangle$. X_1 is perpendicular to itself and therefore has a predecessor. In fact, the predecessors of X_1 are $X_2 = 1+5+7+11+\ldots+(N-1)$ and $X_2 = 2+4+8+\ldots+N$. As $|X_i| = 2n+1$ is odd, $i = 2,3$, we have $\langle X_i, X_1 \rangle = 1$ and neither X_2 nor X_3 has a predecessor. Thus $T(0) =$ co-orb$(0) = \{0, X_1, X_2, X_3\}$ has the form



Any persistent configuration Y therefore must have the following form: $Y = Y_1 \cup Y_2 \cup Y_3$ where $Y_1 \subseteq \{x \in [N] | x \equiv 0 \pmod 3\}$ is arbitrary and both $Y_2 \subseteq X_2$ and $Y_3 \subseteq X_3$ have even cardinality. Hence there are 2^{6n} persistent configurations (this also follows from lemmata 2.3 and 2.4).

Similarly, the kernel of σ^+ over C_N is trivial unless $N \equiv 0 \pmod 3$, in which case it has dimension two and is generated by $Y_1 = 1+2+4+5+\ldots+(N-2)+(N-1)$ and $Y_2 = 2+3+5+6+\ldots+(N-1)+N$; see lemma 4.3 of [4].

Simulations

It is frequently possible to simulate one cellular automaton on another. The evolution of a configuration on the first automaton can thus be studied on the second automaton. As an example, consider P_N and P_{2N+1}, the paths on points $\{1, \ldots, N\}$ and $\{1, \ldots, 2N+1\}$ respectively. Define a map $f(X) := X + \sum_{x \in X} (2N+2-x)$. It is easy to see that $f(\rho(X)) = \rho(f(X))$ where $\rho \in \{\sigma^-, \sigma^+\}$. Hence $\langle P_{2N+1}, \rho \rangle$ simulate $\langle P_N, \rho \rangle$. To make this precise, let us say that the σ-automaton on H *simulates* the σ-automaton on G iff there is a injective linear map $f : C_G \to C_H$ such that for all X in C_G:

$$f(\sigma_G(X)) = \sigma_H(f(X)).$$

The map f will be called a simulation of G on H.

Our next lemma shows that on arbitrary graphs Lindenmayer automata are no less general than σ-automata.

Lemma 2.6. *Every σ-automaton can be simulated by a symmetric automaton with rule σ^+ as well as with rule σ^-.*

Proof. Let $G = \langle V, E\rangle$ be a graph. For the sake of simplicity, we will only show the simulation for σ^+; the argument for σ^- is entirely similar. To simulate the σ-automaton on G we split every vertex of G in two: H has vertices $V \times [2]$. We will write v_i for $(v, i) \in V \times [2]$. For every directed edge $(u, v) \in E$, $u \neq v$, introduce undirected edges $\{u_1, v_i\}$, $i = 1, 2$, in H. Furthermore, introduce an edge $\{v_1, v_2\}$ whenever v has no self loop. Define an injective linear map $f : C_G \to C_H$ by $f(v) := v_1 + v_2$ for all $v \in V$.

Claim $f \circ \sigma_G = \sigma_H^+ \circ f$. By definition,

$$\Gamma_H(v_1) = \{u_1 | u \in \Gamma_G(v), u \neq v\} \cup \{u_1, u_2 | v \in \Gamma_G(u)\}[\cup\{v_2\}]$$

and

$$\Gamma_H(v_2) = \{u_1 | u \in \Gamma_G(v), u \neq v\}[\cup\{v_1\}].$$

The last term $\{v_i\}$ is added iff v has no self loop. But $f(X)$ contains $\{v_1, v_2\}$ iff X contains v. Thus for all configurations X in C_G $f(\sigma_G(X))(v_i)$ $= \sigma_H(f(X))(v)$ and we are done. ∎

Example 3

We will show that $h(P_N, \sigma^+)$ and $h(C_N, \sigma^+)$ have the following form:

$$h(P_N, \sigma^+) = \begin{cases} 0 & N \not\equiv 2 \pmod 3 \\ 2^{o_2(N+1)+1} & N \equiv 2 \pmod 3. \end{cases} \tag{2.1}$$

$$h(C_N, \sigma^+) = \begin{cases} 0 & N \not\equiv 0 \pmod 3 \\ 2^{o_2(N)} & N \equiv 0 \pmod 3. \end{cases} \tag{2.2}$$

Equation (2.2) is stated in [8]. In [3], quotient rings of polynomials are used to derive such results for rules σ^- and σ^+ on cycles. The reference also provides a detailed analysis of the length of the cycles in $\mathcal{C}_{C_N}$. We will show how to derive (2.1) and (2.2) in our framework.

We begin with (2.2). Let $N = 3 \cdot 2^k \cdot m$ where $k \geq 0$ and $m \geq 1$ is odd. Thus $k = o_2(N)$. Define a configuration

$$Y := \sum_{0 \leq i < m} (3 \cdot 2^k \cdot i + 2^k) + (3 \cdot 2^k \cdot i + 2^{k+1}).$$

By a result in [7] $(\sigma^+)^{2^k}(Y) = 0$; in fact, we must have $d(Y) = 2^k$. Also note that Y fails to be perpendicular to Y_1: exactly one of $3 \cdot 2^k \cdot i + 2^k$ and $3 \cdot 2^k \cdot i + 2^{k+1}$ is congruent 1 modulo 3, hence $Y \cap Y_1$ has odd cardinality and $\langle Y, Y_1 \rangle = 1$. By theorem 2.4, Y has no predecessor; thus, Y is a leaf of $T(0)$. By lemma 2.3 and the remark following it $T(0)$ is a balanced tree of height $h(C_N, \sigma^+)$. Hence $2^k = h(Y) = h(C_N, \sigma^+)$ and we are done.

To establish (2.1) we use a simulation of P_N by C_{2n+2}. The appropriate map in this case is

$$f(X) := X + \sum_{x \in X} (2N + 2 - x).$$

Using the symmetry properties of σ^+, it is easy to verify that f is indeed a simulation. We may safely assume that $N = 2 + 3 \cdot 2^k \cdot n$ where $k \geq 0$ and $n \geq 1$ is odd. Note that the configuration 1 on P_N is not perpendicular to X_1, thus 1 fails to have a predecessor and must be a leaf in the transition diagram $\mathcal{C}_{P_N}$ By lemma 2.4 and proposition 2.5 we have $h(P_N, \sigma^+) = h(1; P_N)$. But $f(1) = 1 + (2N + 1)$ and clearly $h(1 + (2N + 1); C_{2N+2}) = h(1 + 3; C_{2N+2})$. As configuration $1 + 3$ is not perpendicular to Y_1, it must be a leaf in $\mathcal{C}_{C_{2N+N}}$. As before, we can argue that $h(1 + 3; C_{2N+2}) = h(C_{2N+2})$. Now $h(C_{2N+2})$ $= 2^{o_2(2N+2)} = 2^{o_2(N+1)+1}$ by (2.2); thus (2.1) follows are we are done.

Infinite Automata

Let us briefly consider automata $A = \langle G, \sigma \rangle$ where the underlying graph G is infinite. Recall that G is always assumed to be locally finite. The behavior of the automaton A is completely determined by the connected components of the underlying graph G: for a connected component H of G and any configuration X we have $\sigma_G(X) \cap H = \sigma_H(X \cap H)$. The connected components of a locally finite graph are all countable, so we may safely assume that G is countably infinite. C_G can be construed as a topological space: endow $F_2 = \{0, 1\}$ with the discrete topology and $C_G = \prod_V F_2$ with the corresponding product topology. As V is countable, the resulting space is homeomorphic to the Cantor space 2^ω. 2^ω is well known to be a compact Hausdorff space. For any convergent sequence $Y_i : i < \omega)$ in C_G we write $\lim_{i \to \infty} Y_i$ for the limit with respect to this topology. Rule σ_G is certainly continuous in this topology. The next lemma allows us to lift results from finite to infinite graphs.

Lemma 2.7. Extension lemma. *Let $(Y_i : i < \omega)$ be a sequence of configurations in G such that $Y = \lim_{i \to \infty} Y_i$. Suppose for all $i \geq 0$ there exists a configuration X_i such that $Y_i := \sigma(X_i)$. Then there is a subsequence $(X_{i_j} : j < \omega)$ such that $X := \lim_{j \to \infty} X_{i_j}$ exists and $\sigma(X) = Y$.*

Proof. C_G is a compact Hausdorff space; hence the infinite sequence $(X_i :$ $i < \omega)$ must possess a limit point X and a subsequence $X_{i_j} : i < \omega)$ that converges to X. But then by the of $\sigma_G : \sigma(X) = \sigma(\lim_{j \to \infty} X_{i_j}) = \lim_{j \to \infty} \sigma(X_{i_j})$ $= \lim_{j \to \infty} Y_{i_j} = Y$. ■

Note that the only property of σ used in the last proof is its continuity. Hence the extension lemma holds for arbitrary cellular automata rather than just σ-automata.

The typical application of the extension lemma is as follows. Suppose $G = \langle V, E\rangle$ is the limit of an ascending chain of finite subgraphs G_i, $i \geq 0$. To be more explicit, suppose $G_i = \langle V_i, E_i\rangle$ where $V_i \subseteq V_{i+1} \subseteq V$ is finite, $E_i \subseteq E_{i+1} \subseteq E$ and $V = \bigcup_{i\geq 0} V_i$, $E = \bigcup_{i \geq 0} E_i$. If the σ-automata $\langle G_i, \sigma\rangle$ are all reversible, then the $\langle G, \sigma\rangle$ is complete. For let Y be an arbitrary configuration on G and let X_i be the predecessor of $Y_i' := Y \cup V_i$ in G_i. Set $Y_i := \sigma_G(X_i)$. As G is locally finite, we must have $\lim_{i\to\infty} Y_i = Y$. Hence by the extension lemma there is some configuration X such that $\sigma_G(X) = Y$. On the other hand, suppose there is a non-trivial predecessor X_i of 0 in G_i for all $i \geq 0$ such that $\lim_{i\to\infty} |X_i| = \infty$. Letting $Y_i := \sigma_G(X_i)$ one has $\lim_{i\to\infty}$ $Y_i = 0$. Hence by the extension lemma there is a subsequence $(X_{i_j} : i < \omega)$ that converges to some configuration X such that $\sigma_G(X) = 0$. As $\lim_{i\to\infty} |X_i|$ $= \infty$ X is non-trivial; hence G is irreversible.

Accordingly by (2.1) and an analogous result for rule σ^-, both Lindenmayer automata on the bi-infinite path P_∞ are complete and irreversible. The kernel of σ^+ has dimension two and is generated by the configurations

$$X_0 := \sum_{i\in Z} 3i + (3i-1) \text{ and } X_1 := \sum_{i \in Z}(3i+1) + (3i-1).$$

The kernel of σ^- also has dimension two and is generated by the configurations

$$X_0 := \sum_{i\in Z} 2i \text{ and } X_1 := \sum_{i\in Z}(2i+1).$$

We note in passing that Lindenmayer automata on finite graphs with rule σ^+ have the property that the all-ones configuration 1 always has a predecessor (this follows easily from corollary 2.5). A basis for the affine subspace $(\sigma^+)^{-1}(1)$ can be computed in polynomial time by solving the system of equations $(E^T + I) \cdot X = 1$. However, it is NP-hard to find the solution of minimal cardinality. For a proof, see [4,5]. By the extension lemma configuration 1 has a predecessor in all Lindenmayer automata with rule σ^+, finite or infinite.

3. Lindenmayer automata on product graphs

In this last section, we will study the reversibility of σ-automata on finite product graphs such as grids and cylinders. We will focus on Lindenmayer automata, though some of the results hold for σ-automata in general. Throughout this section assume that $G = \langle V, E\rangle$ is a finite graph and define

$$d(G) := \dim(\ker(\sigma_G)) = \log_2(|\ker(\sigma_G)|).$$

Thus $d(G)$ is the co-rank of σ_G as a linear map from the configurations space to itself. $d(G)$ measures the degree of reversibility of $A = \langle G, \sigma\rangle$: the σ-automaton on G is reversible iff $d(G) = 0$. As G is finite, this is equivalent with σ_G begin surjective, i.e. every configuration has a predecessor iff every configuration has at most one predecessor iff $d(G) = 0$. It is straightforward to compute $d(G)$ from the adjacency matrix of G. However, no structural properties of the underlying graphs are known that characterize $d(G)$. Even for Lindenmayer automata on regular graphs, no concise description of $d(G)$ is available. The somewhat easier question of whether a graph G is reversible under rule σ also turns out to be difficult to answer. Decision procedures that determine reversibility for trees (connected acyclic undirected graphs) are given in [1] and [4]. The second reference contains a list of reduction procedures on undirected graphs (deletions of edges and/or vertices) that preserve reversibility. A simple example of such a procedure is the deletion of double end points. Suppose u_1 and u_2 are two end points (i.e. vertices of degree 1), both adjacent to vertex v. Then the graph G' obtained from G by deleting u_1 and u_2 is reversible iff G is reversible.

For certain simple graphs we will be able to determine the co-rank of σ explicitly. For example, rule σ^- on a $m \times m$ square grid has co-rank m, $d(P_{m,m}, \sigma^-) = m$. Unfortunately, the situation for rule σ^+ is much more complicated. Table 1 lists $d(P_{m,m}, \sigma^+)$ for $m \leq 100$. Figure 1 shows the irreversible rectangular $m \times n$ grids and cylinders for $1 \leq n, m \leq 40$ and rule σ^+.

We will focus on graphs with strong symmetry properties such as grids, cylinders, and tori. The main tools in studying the reversibility of these graphs are symmetries and simulations as defined previously. Note that if the σ-automaton on G_1 is irreversible and can be simulated on G_2, then G_2 is also irreversible. In fact, $d(G_2) \geq d(G_1)$ (recall that a simulation is required to be injective).

We begin with a general definition of product graphs suitable for our purposes. Let $G = \langle V, E \rangle$ be an arbitrary graph and $n \geq 1$ a number. Define the *acyclic product graph* $G \times [n]$ as follows: $G \times [n]$ has vertex set $V \times [n]$ and edges

$$E_n := \{((u,i),(v,j)) | (i = j \wedge (u,v) \in E) \vee (u = v \wedge |i - j| = 1)\}.$$

Similarly, the *cyclic product graph* $G \times (n)$ has vertex set $V \times [n]$ and edge set

$$\begin{aligned} E'_n \quad &:= \{((u,i),(v,j)) | (i = j \wedge (u,v) \in E) \\ &\vee (u = v \wedge (|i - j| = 1 \vee \{i,j\} = \{1,n\}))\}. \end{aligned}$$

The most important examples are rectangular grids $P_{m,n} = P_m \times [n]$, cylinders $C_{m,n} = C_m \times [n]$, and tori $T_{m,n} = C_m \times (n)$. Note that cylinder $C_{m,n}$ is isomorphic to $P_n \times (m)$. We will denote the infinite two-dimensional grid by P^2_∞ (P^2_∞ has vertices $Z \times Z$ and there is an edge $\{(x,y),(x',y')\}$ iff $|x - x'| + |y - y'| = 1$).

Here are some notational conventions. We will write v^i instead of $(v,i) \in V \times [n]$. For a configuration X in a product graph, let $X^i := X \cap V \times \{i\}$ be the "i^{th} row" of X, $i \in [n]$.

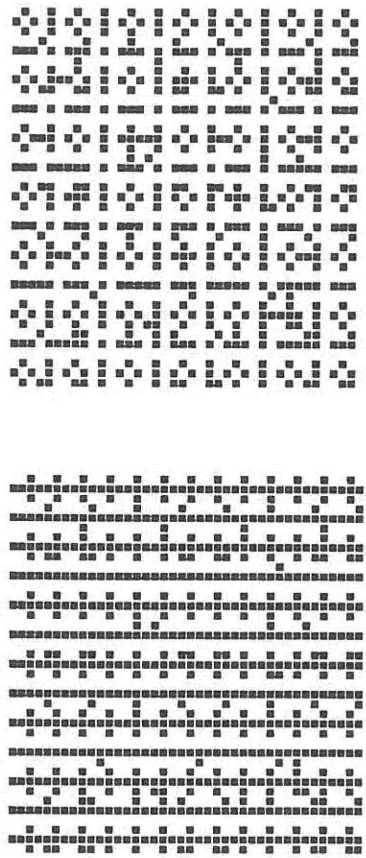


Figure 1: Irreversible grids $P_m \times P_n$ (top) and cylinders $C_m \times P_n$ (bottom) under rule σ^+. A box in position m, n indicates that $P_m \times P_n$ (respectively $C_m \times P_n$) is irreversible. $m = 1, \ldots, 40 \rightarrow n = 1, \ldots, 40 \downarrow$.

m	$d(P_{m,m},\sigma^+)$	m	$d(P_{m,m},\sigma^+)$
4	4	53	2
5	2	54	4
9	8	59	22
11	6	61	40
14	4	62	24
16	8	64	28
17	2	65	42
19	16	67	32
23	14	69	8
24	4	71	14
29	10	74	4
30	20	77	2
32	20	79	64
33	16	83	6
34	4	84	12
35	6	89	10
39	32	92	20
41	2	94	4
44	4	95	62
47	30	98	20
49	8	99	16
50	8		

Table 1: Irreversible Lindenmayer automata on square grids $P_{m,m}, m \leq 100$.

Proposition 3.1. *The σ-automaton on $G\times[n]$ can be simulated by $G\times[2n+1]$. Hence $d(G\times[n]) \leq d(G\times[2n+1])$ and the σ-automaton on $G\times[2n+1]$ is reversible only if the σ-automaton on $G\times[n]$ is reversible. If, in addition, G is reversible, then $G\times[2n+1]$ is reversible iff $G\times[n]$ is reversible.*

Proof. The map $f : C_{G\times[n]} \to C_{G\times[2n+1]}$ defined by

$$f(X)(x^i) := \begin{cases} 0 & i = n+1 \\ X(x^j) & j = i \text{ or } j = i-n-1,\ j \in [n]. \end{cases} \tag{3.1}$$

is a simulation. This follows easily from the symmetry properties of σ. Hence $\ker(\sigma_{G\times[2n+1]}) \supseteq f(\ker(\sigma_{G\times[n]}))$ and the first claim follows. If the kernel of σ on $G\times[n]$ is non-trivial, pick a configuration $X \neq 0$ in the kernel. Define a configuration Y in C_H by $Y = (X^1,\ldots,X^n,0,X^n,\ldots,X^1)$. Similarly, $\sigma_H(Y)(v^i) = \sigma_H(Y)(v^{2n+2-i}) = \sigma_{H'}(X)(v^i) = 0$ for all v in V, $i \in [n]$. Hence $\sigma_H(Y) = 0$ and $Y \neq 0$ as required.

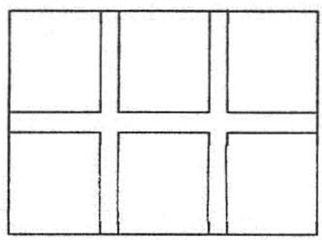
On the other hand, suppose the kernel of σ on $G\times[2n+1]$ is non-trivial and G is reversible. By proposition 3.1 there is a configuration $Y \neq 0$ in the kernel of $\sigma_{G\times[2n+1]}$ such that $F_h(Y) = Y$. Note that for all v in V $Y(v^n) = Y(v^{n+2})$. As $\Gamma_{G\times[2n+1]}(v^{n+1}) = \Gamma_G(v)\times\{n+1\} \cup\{v^n, v^{n+2}\}$ this implies $\sigma_{G\times[2n+1]}(Y)(v^{n+1}) = 0$. Hence $\tau(Y^{n+1}) = 0$ and $Y^{n+1} = 0$. Set $X := Y \cap V \times [n]$. Then $X \neq 0$ is in the kernel of $\sigma_{G\times[n]}$; hence $G \times [n]$ is irreversible and we are done. ■

It follows from proposition 3.1 by induction on k that every $(2^k-1)\times(2^k-1)$ square grid is reversible under rule σ^+: G is here the path on 2^k-1 points and therefore reversible (as $2^k-1 \not\equiv 2 \pmod 3$). In fact, any r-dimensional hypercube of the form

$$(2^{k_1}-1)\times(2^{k_2}-1)\times\ldots\times(2^{k_r}-1)$$

is reversible under rule σ^+. As we will see below there are d-dimensional grids of arbitrary size that fail to be reversible under rule σ^+ and have predecessors of 0 of arbitrary size. Hence by the extension lemma, the infinite d-dimensional grid is both complete and irreversible (the dimension of the kernel is 2^0). The same holds for rule σ^-.

Simulations as in the last proof can be used frequently to obtain lower bounds on the co-rank of σ on product graphs. For example, suppose that $p+1$ divides $m+1$ and $q+1$ divides $n+1$. Then $P_{p,q}$ can also be simulated by $P_{m,n}$. To see this, first note that the $m \times n$ rectangle can be covered with $p \times q$ rectangles in such a way that the smaller rectangles are separated by gaps of width 1 as shown below.



Now define a map $f : C_{P_{p,q}} \to C_{P_{m,n}}$ by:

$$f((x,y)) := \{(x_i + 1\cdot(p+1), y_j + j\cdot(p+1)) | 0 \leq i < m_0, 0 \leq j < n_0\}$$

where $m_0 := (m+1)/(p+1)$, $n_0 := (m+1)/(q+1)$ and

$$x_i := \begin{cases} a & i \text{ even} \\ p - x + 1 & i \text{ odd} \end{cases}$$

and similarly for y_j. Thus, f places one copy of configuration X–possibly after a horizontal or vertical reflection–into each of the $p \times q$ rectangles. It is straightforward to show that f is indeed a simulation. Hence $d(P_{p,q}) \leq d(P_{m,n})$.

Similarly the torus $T_{p,q}$ can be simulated by the torus $T_{m,n}$ whenever p divides m and q divides n, and the cylinder $C_{p,q}$ can be simulated by the cylinder $C_{m,n}$ whenever p divides m and $q+1$ divides $n+1$. Any grid, cylinder, and torus can be simulated by the infinite square grid P^2_∞. Furthermore, the subspace of all configurations on P^2_∞ invariant under a right shift by m and an up shift by n units has a simulation in the torus $T_{m,n}$.

Extending configurations

Geometric arguments as above do not allow to determine the co-rank of σ. In the following, we will outline an algebraic approach that provides some more information about the dimension of the kernel of linear rules on product graphs. To describe the co-rank of $\sigma_{G\times[n]}$, first note that a configuration X in the kernel of $\sigma_{G\times[n]}$ is completely determined by its first row X^1 : $X^2 = \tau(X^1)$, $X^3 = \tau(X^2) + X^1$ and so forth. On the other hand, suppose X_0 is an arbitrary configuration on G. Inductively define a sequence of configurations on G by

$$X_1 := \tau(X_0)$$

$$X_{i+2} := \tau(X_{i+1}) + X_i.$$

X_n is a linear function of X_0 for all $n \geq 0$. To make this more obvious, inductively define a sequence of polynomials π_n, $n \geq 0$, in the polynomial ring $F_2[\tau]$.

$$\pi_0 := \mathrm{id}$$

$$\pi_1 := \tau$$

$$\pi_{i+2} := \tau \circ \pi_{i+1} + \pi_i. \tag{3.2}$$

Thus e.g. $\pi_{25} = \tau^1 + \tau^5 + \tau^{17} + \tau^{21} + \tau^{25}$. The substitution $\tau \mapsto \sigma_G$ induces a ring homomorphism $h : F_2[\tau] \to (C_G \to C_G)$. We will write $\pi_n[\sigma_G]$ for the image of π_n under this h. Then $\pi_n[\sigma_G](X_0) = X_n$. The $m \times m$ matrices over F_2 representing $\pi_{25}[\sigma_{P_{25}}]$ and $\pi_{50}[\sigma_{P_{50}}]$ are shown in figure 2.

Now define an extension map $\mathrm{ext}_n : C_G \to C_{G\times[n]}$ by $\mathrm{ext}_n(Z) = \sum_{i\in[n]} \pi_i(Z) \times \{i+1\}$. It is easy to see that $\sigma(\mathrm{ext}_n(Z))(v^i) = 0$ for all $v \in V$, $1 \leq i < n$. Hence we have

$$\mathrm{ext}_n(Z) \text{ lies in the kernel of } \sigma_{G\times[n]} \text{ iff } \pi_n[\sigma_G](Z) = 0. \tag{3.3}$$

According to (3.3), the configuration Z in C_G can be extended to at most one configuration in e kernel of $\sigma_{G\times[n]}$, namely to $\mathrm{ext}_n(Z)$. This is the case iff $\pi_n[\sigma_G](Z) = 0$. By the linearity of ext_n and π_n, we have established the following lemma which provides an upper bound on the co-rank of rule σ on $G \times [n]$.

Lemma 3.2. *The kernel of $\sigma_{G\times[n]}$ has the form $\mathrm{ext}_n(\ker(\pi_n[\sigma_G]))$. In particular, $d(G \times [n]) =$ co-rank$(\pi_n[\sigma_G]) \leq |G|$.*

The situation for cyclic product graphs of the form $G \times (n)$ is quite similar. Suppose X_0 and X_1 are two configurations on G. As before for acyclic product automata, define inductively a sequence of configurations by

$$X_{i+2} := \tau(X_{i+1}) + X_i.$$

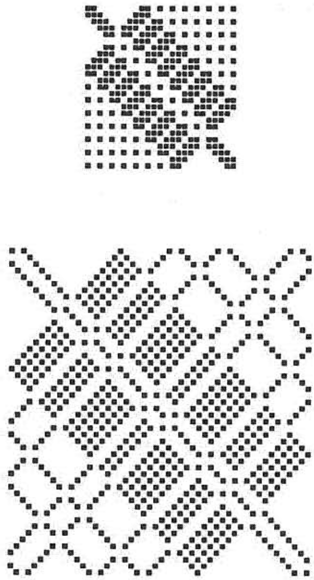


Figure 2: The F_2-matrices representing $\pi_{25}[\sigma^+_{P_{25}}]$ and $\pi_{50}[\sigma^+_{P_{50}}]$. A box represents a 1 and a blank represents a 0 in F_2.

Using the linearity of τ one verifies by induction that for all $i \geq 0$

$$X_i := \pi_{i-2}(X_0) + \pi_{i-1}(X_1).$$

Here, we assume $\pi_{-1} := 0$ and $\pi_{-2} := \text{id}$. Therefore, define $\Pi_n : C_G \times C_G \to C_G \times C_G$ by

$$\Pi_n(X,Y) := (\pi_{n-2}(X) + \pi_{n-1}(Y),\ \pi_{n-1}(X) + \pi_n(Y)). \tag{3.4}$$

Again, define an extension map $\text{ext}_n : C_G \times C_G \to C_{G\times(n)}$ by $\text{ext}(X,Y) := \sum_{i\in[n]}(\pi_{i-2}(X) + \pi_{i-1}(Y)) \times \{i+1\}$. It is not hard to see that X_0, X_1 can be extended to a configuration in the kernel of σ on $G \times (n)$ iff

$$\begin{aligned} \text{ext}_n(X_0, X_1) \text{ lies in the kernel of } \sigma \text{ iff} \\ X_n = X_0 \text{ and } X_{n+1} = X_1 \text{ iff} \\ \Pi_n(X_0, X_1) = (X_0, X_1). \end{aligned} \tag{3.5}$$

We have the following analogue of lemma 3.2.

Lemma 3.3. *The kernel of $\sigma_{G\times(n)}$ has the form $\text{ext}_n(\ker(\Pi_n[\sigma_G]))$. In particular, $d(G \times (n)) = \text{co-rank}(\Pi_n[\sigma_G]) \leq 2 \cdot |G|$.*

For some product graphs, the extension procedure from the last lemmata can be used to explicitly construct predecessors of 0 in $G \times [n]$. The next proposition is easily established by induction on n.

Proposition 3.4. *Let X in C_G be a fixed point of σ_G and Y in the kernel of σ_G. Then*

$$\pi_n[\sigma_G](X) = \begin{cases} X & n \not\equiv 2 \pmod 3 \\ 0 & n \equiv 2 \pmod 3. \end{cases} \tag{3.6}$$

$$\pi_n[\sigma_G](Y) = \begin{cases} Y & n \text{ even} \\ 0 & n \text{ odd.} \end{cases} \tag{3.7}$$

Example 4

In the special case $G = C_m$ there is always a non-trivial fixed point of $\sigma_G : \sigma_G(1) = 1$. Hence, by (3.5), if $n \equiv 2 \pmod 3$, the configuration $(1,1,0,1,1,0,\ldots,1,1)$ on the cylinder $C_{m,n}$ lies in the kernel of σ^+. If 3 divides m and n is odd, then by (3.6) the configuration $(Y,0,Y,0,\ldots,0,Y)$ lies in the kernel of σ^+ on the cylinder $C_{m,n}$ where $Y = 1+2+4+5+\ldots+(m-2)+(m-1)$. Similarly, if $n \equiv 0 \pmod 3$, the configuration $(1,1,0,1,1,0,\ldots,1,1)$ on the torus $T_{m,n}$ lies in the kernel of σ^+ by (3.4). If 3 divides m and n is even, then the configuration $(Y,0,Y,0,\ldots,0,Y)$ lies in the kernel of σ^+ on the torus $T_{m,n}$.

Second order σ-automata

The sequence of configurations $\pi_i(Z)$ in C_G, $i \geq 0$, in the extension procedure can also be thought of as the evolution of Z on a second-order σ-automaton on G. In second-order σ-automaton, the next configuration depends not only on the current configuration but also its predecessor. Initially, two seed configurations are needed to begin the evolution. In particular, a second-order σ-automaton is a graph $G = \langle V, E_1, E_2\rangle$ with two edge sets E_1 and E_2. Let $\sigma_i : C_G \to C_G$ denote the rule determined by E_i (i.e., $\sigma_i(X) = E_i^T \cdot X$) and define the second-order rule $\sigma_{[2]} : C_G \times C_G \to C_G$ by

$$\sigma_{[2]}(X,Y) := \sigma_1(X) + \sigma_2(Y).$$

Thus, $\sigma_{[2]}(X,Y)$ is a linear function of both X and Y. Given seed configurations X_0, X_1 one may inductively define a sequence of configurations by

$$X_n := \sigma_{[2]}^n(X_0, X_1) := \sigma_{[2]}(X_{n-2}, X_{n-1})$$

for all $n \geq 2$.

Specifically, to generate the sequence of configurations that occurs during the extension of a seed configuration Z in C_G, define the second-order rule $\underline{\sigma} : C_G^2 \to C_G$ by

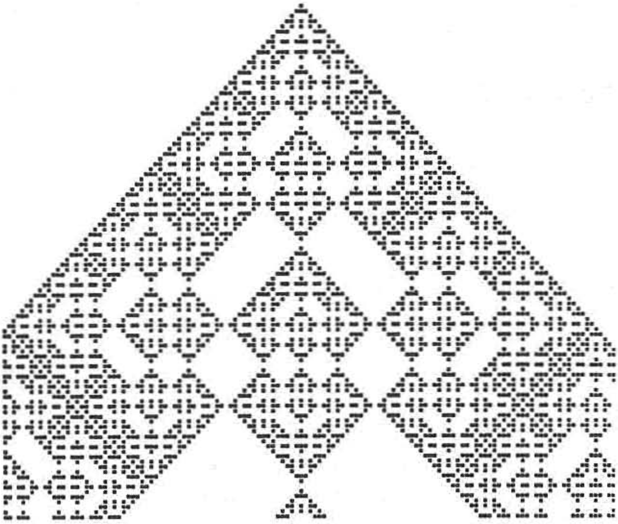


Figure 3: Evolution of initial configurations 0, $\sigma^+(0)$ on P_∞ under rule $\underline{\sigma}^+$

$$\underline{\sigma}(X,Y) := X + \sigma_G(Y).$$

Clearly, $\pi_i(Z) = \underline{\sigma}^i(Z, \sigma_G(Z))$ for all $i \geq 0$. For undirected graphs $\underline{\sigma}^+$ and $\underline{\sigma}^-$ are defined in the obvious fashion.

Figure 3 shows the first 100 generations obtained from seed configurations 0, $\sigma^+(0)$ on the bi-infinite path P_∞ using rule $\underline{\sigma}^+$. The two-dimensional pattern obtained in this fashion is self-similar and has fractal dimension $\log_2((3+\sqrt{17})/2 \approx 1.83$. Note, however, that rule $\underline{\sigma}^-$ generates a simple regular checkerboard pattern (see also lemma 3.7 below).

The following lemma describes the periodicity properties of $\underline{\sigma}$-automata.

Lemma 3.5. *Let S be a finite dimensional vector space over F_2 and $\tau : S \to S$ a linear operator on S, $m \geq 1$. Define the polynomials π_n as in (3.2). Then there exists a number $N \geq 2$ such that*

(1) $\pi_N[\tau] = 0$ *and* $\pi_{N+1}[\tau] = id$.

(2) *For all* $n \geq 0; \pi_n[\tau] = \tau_{m \bmod N+1}[\tau]$.

(3) *For all* $k, 0 \leq k \leq N : \pi_{k-1}[\tau] = \pi_{N-k}[\tau]$ *(3.1)*

Proof. Let $\mathrm{End}(S)$ denote the ring of all linear maps from S to itself. Define a function $\rho : \mathrm{End}(S)^2 \to \mathrm{End}(S)^2$ by $\rho(f,g) := (g, \tau \circ g + f)$. Note that ρ

is a bi-linear map. Furthermore, ρ is clearly injective and thus a bijection. Thus, for every pair (f,g) in $\mathrm{End}(S)^2$ there exists a number $r \geq 0$ such that $\rho^r(f,g) = (f,g)$. Let $r(f,g)$ denote the least such r. For $\tau \neq 0$, we must have $r(\mathrm{id},\tau) \geq 2$. Furthermore, for all $f,g : \rho^{-1}(f,g) = (\tau \circ f + g, f)$. Hence for $\tau \neq 0$ $\rho^{r(\mathrm{id},\tau)-1}(\mathrm{id},\tau) = \rho^{-1}(\mathrm{id},\tau) = (0,\mathrm{id})$. But $\rho^n(\mathrm{id},\tau) = (\pi_n[\tau], \pi_{n+1}[\tau])$. Setting $N := r(\mathrm{id},\tau) - 1$ we have $\pi_N[\tau] = 0$ and $\pi_{N+1}[\tau] = \mathrm{id}$; thus, (1) and (2) follow. A straightforward induction on k now establishes (3). ∎

Following lemma 3.5 we may define

$$\gamma(G) := \min(n \geq 1 | \pi_n[\sigma_G] = 0 \wedge \pi_{n-1}[\sigma_G] = \mathrm{id}$$

and

$$\overline{\gamma}(G) := \min(n \geq 1 | \pi_n[\sigma_G] = 0).$$

For symmetric graphs G, the functions $\gamma(G,\sigma^+)$, $\gamma(G,\sigma^-)$ and so forth are defined analogously. It is convenient to think of the sequence $(\pi_i[\tau] : i \geq 0)$ as being extended to $(\pi_i[\tau] : i \geq -1)$ where $\pi_{-1}[\tau] := 0$. The latter sequence consists of infinitely many repetitions of the following basic block (for the sake of clarity we write π_i instead of $\pi_i[\tau]$):

$$\pi_{-1}, \pi_0, \ldots, \pi_k, \pi_k, \ldots, \pi_0 \qquad \text{for } \gamma(G) \text{ even }, k = \gamma(G)/2 \text{ and}$$

$$\pi_{-1}, \pi_0, \ldots, \pi_{k-1}, \pi_k, \pi_{k-1}, \ldots, \pi_0 \qquad \text{for } \gamma(G) \text{ odd}, k = \lfloor \gamma(G)/2 \rfloor. \quad (3.2)$$

Hence the sequence $(\pi_i[\tau] : i \geq -1)$ has a period of length $\gamma(G) + 1$. Consequently, $d(G \times [n]) = d(G \times [n_0])$ where $n_0 := n \bmod \gamma(G) + 1$. Now choose configurations $X_1, \ldots, X_d$, $d := d(G \times [n_0])$, that can be extended to a basis of the kernel of σ on P_{n_0}. Then a basis for the kernel of σ on $G \times [n]$, $n \equiv n_0 \pmod{\gamma(G) + 1}$, has the form $\mathrm{ext}_n(X_i)$, $i = 1, \ldots, d$. Then configurations in the basis consist of several copies of the configurations $\mathrm{ext}_{n_0}(X_i)$.

One can show that all the terms $\pi_i[\tau]$ are different from 0, with the possible exception of $\pi_k[\tau]$ in the second case. Hence $\overline{\gamma}(G) < \gamma(G)$ implies $\gamma(G) = 2\overline{\gamma}(G) + 1$.

Theorem 3.6. *For every finite graph G there exists a number $n \geq 1$ such that (1) the kernel of rule σ on the product graph $G \times [n]$ has dimension $|G|$, (2) the kernel of rule σ on the cylinder $G \times (n+1)$ has dimension $2 \cdot |G|$.*

Proof. According to lemma 3.5, we can set $n := \gamma(G)$ for the first part of the theorem. To maximize the co-rank of σ for cyclic products of the form $G \times (m)$, we have to make sure that $\Pi_m(X,Y) = (X,Y)$ for all X,Y in C_G. According to (3.4) and (3.5), it suffices to have $\pi_{m-1} = 0$ and $\pi_{m-2} = \pi_m = \mathrm{id}$. But $m = \gamma(G) + 1$ has these properties again by lemma 3.5 and we are done. ∎

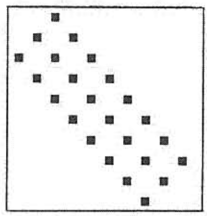
In order to determine $\gamma(P_m, \sigma^-)$ and $\gamma(C_m, \sigma^-)$ let us define the following "checkerboard" matrices over F_2. Let $m \geq 1$ and for k, $1 \leq k \leq m$, define matrices $M_{k,m}$ in $F_2^{m,m}$ by $M_{k,m}(i,j) = 1$ iff

$j = k - 1 - i + 2v$, some $v = 0, \dots, i-1, i \leq k$ or

$j = 1 - k + i + 2v$, some $v = 0, \dots, k-1, k \leq i \leq m - k + 1$ or

$j = 1 - k + i + 2v$, some $v = 0, \dots, m-i, m-k+1 \leq i \leq m$.

The following picture shows $M_{3,10}$.



Observe that $M_{k,m}$ when construed as a configuration on $P_m \times P_m$ is a predecessor of 0. In fact, $\{M_{k,m} | k \in [m]\}$ is a basis of the kernel of σ^-. The cardinality of $M_{k,m}$ as a configuration is $k \cdot (m - k + 1)$. Hence, by theorem 2.4, 1 has a predecessor on $P_m \times P_m$ under rule σ^- iff m is odd.

Returning to the function $\gamma(P_m, \sigma^-)$ note that $M_{2,m}$ is the matrix representation of $\sigma^-_{P_m}$. By induction on n one can easily show that $M_{k,m} \cdot M_{2,m}$ $+ M_{k-1,m} = M_{k+1,m}$ and $M_{k,m} \cdot M_{2,m} + M_{k+1,m} = M_{k-1,m}$ for all appropriate k. Thus, we have established the following lemma.

Lemma 3.7. *Let $m \geq 1$ and let P_m be the path on m points. Then the matrix over F_2 representing $\pi_n[\sigma^-_{P_m}]$ has the following form:*

$$\pi_n[\sigma^-_{P_m}] = \begin{cases} 0 & \text{if } n \equiv m, 2m+1 (\mathrm{mod} 2m+2) \\ M_{k,m} & \text{if } k = 1 + n \bmod (2m+2) \\ M_{k,m} & \text{if } k = 2m + 1 - n \bmod (2m+2). \end{cases} \tag{3.3}$$

Thus, $\gamma(P_m, \sigma^-) = 2 \cdot m + 1$ and $\overline{\gamma}(P_m, \sigma^-) = m$.

Lemma 3.8. *Let $m \geq 1$ and let C_m be the cycle on m points. Then*

$$\gamma(C_m, \sigma^-) = \begin{cases} m - 1 & m \text{ even} \\ 2m - 1 & m \text{ odd}. \end{cases} \tag{3.4}$$

Furthermore, $\overline{\gamma}(C_m, \sigma^-) = \gamma(C_m, \sigma^-)$ for all m.

Proof. First consider the Lindenmayer automaton with rule σ^- on P_∞. A simple induction shows that $\pi_t[\sigma^-_{P_\infty}](0) = \sum_{i-0,\dots,t}(-t + 2i)$. Now let C_m be a cycle on points $\{0, \dots, m-1\}$. Clearly, $\pi_t[\sigma^-_{C_m}](0) = \sum_{i=0,\dots,t}(-t + 2i)$ mod$m$. Let us assume m is even, say $m = 2k$. Then certainly $\pi_t[\sigma^-_{C_m}](0)$

$\neq 0$ for all $t < m-1$. But $\pi_{m-1}[\sigma^-_{C_m}](0) = \sum_{i=0,\dots,2k-1}(-2k+1+2i) \bmod 2k$ $= \sum_{i=0,\dots,k-1} 1+2i + \sum_{i=k,\dots,2k-1} -2k+1+2i = \sum_{i=0,\dots,k-1} 1+2i + \sum_{i=0,\dots,k-1} 1+2i = 0$. Thus, $\gamma(C_m, \sigma^-) = \overline{\gamma}(C_m, \sigma^-) = m-1$. The argument for odd m is similar and will be omitted. ∎

Our next theorem gives a closed form description for $d(P_m \times P_n, \sigma^-)$. For natural numbers x, y let $\gcd(x, y)$ denote their greatest common divisor.

Theorem 3.9. *For all m, $n \geq 0$ the kernel of rule σ^- on the $m \times n$ grid $P_m \times P_n$ has dimension $\gcd(m+1, n+1) - 1$. In particular, the $m \times n$ grid is reversible under rule σ^- iff $m+1$ and $n+1$ are relatively prime.*

Proof. For the sake of simplicity, let us write $[m,n]$ for $d(P_m \times P_n, \sigma^-)$. Thus for example $[m,n] = [n,m]$. Using this fact as well as the periodicity of the π_n operators one obtains the following recurrence relations.

$$[m,n] := \begin{cases} 0 & \text{if } mn = 0 \\ m & \text{if } n = m \vee n = 2m+1 \\ [n,m] & \text{if } n < m \\ [m, 2m-n] & \text{if } m < n \leq 2m \\ [m, n \bmod (2m+2)] & \text{if } n \geq 2m+2. \end{cases} \tag{3.5}$$

Equation (3.12) may be construed as a recursive algorithm for the computation of $d(P_m \times P_m, \sigma^-)$. Notice that the algorithm is vaguely similar to the Euclidean algorithm for the greatest common divisor of two numbers. Correctness of the algorithm is established by induction on the depth of the recursion. Clauses 1 and 3 are trivial. Clauses 2, 4, and 5 all follow from (3.9) together with lemma 3.7. To see convergence, observe that the value of $m+n$ decreases at least at every other step in the recursion.

Finally, one shows by induction on the depth of the recursion in (3.12) that $[m,n] = \gcd(m+1, n+1) - 1$. This is obvious for clauses 1, 2, and 3. For the fourth clause, we have $[m,n] = [m, 2m-n] = \gcd(m+1, 2m-n+1) - 1 = \gcd(m+1, 2(m+1) - (n+1)) - 1 = \gcd(m+1, n+1) - 1$. Similarly, for the fifth clause $[m,n] = [m, n \bmod 2m+2] = \gcd(m+1, (n \bmod 2m+2) + 1) - 1 = \gcd(m+1, (n \bmod 2(m+1) + 1) - 1 = \gcd(m+1, n+1) - 1$.

This finishes the proof. ∎

As an immediate consequence of theorem 3.9, all grids of the form $P_{p-1} \times P_n$ where p is a prime are irreversible under rule σ^- iff $n+1$ is a multiple of p, in which case the dimension of the kernel of σ^- has dimension $p-1$. Figure 4 shows the irreversible grids $P_m \times P_n$ fr $1 \leq n$, $m \leq 80$. Geometrically the last result may be interpreted as follows. Let $p := \gcd(m+1, n+1)$. One can define a simulation $f : P_{p,p} \to P_{m,n}$ as in the remark following proposition 3.1. As $\overline{\gamma}(P_p, \sigma^-) = p$ f lifts all the 2^p configurations in the kernel of σ^- on $P_{p,p}$ to configurations in the kernel of σ^- on $P_{m,n}$. Hence $d(P_{p,p}, \sigma^-) \leq d(P_{m,n}, \sigma^-)$ and the theorem shows that equality holds. Hence the kernel of $\sigma^-_{P_{m,n}}$ has the form $f(\ker(\sigma^-_{P_{p,p}}$. By theorem 2.7, the configuration 1 has a predecessor on $P_{m,n}$ under rule σ^- iff p is even or $(m+1)(n+1)/(p+1)^2$ is even.

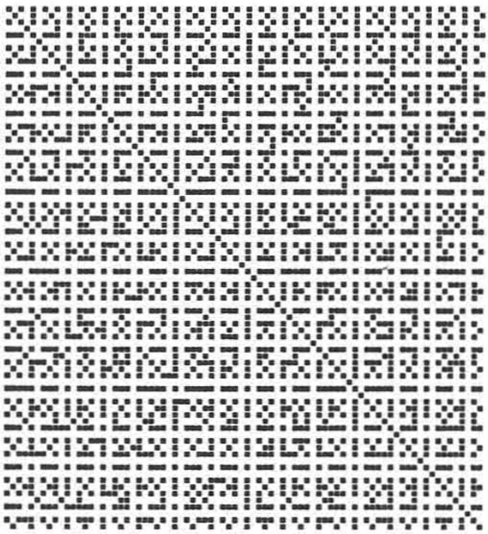


Figure 4: Irreversible grids $P_m \times P_n$ under rule σ^-, $1 \leq m, n \leq 80$.
A box in position m, n indicates that $P_m \times P_n$ is irreversible.

The proof of theorem 3.9 can easily be modified to obtain a corresponding result for tori of the form $C_m \times C_m$. With slightly more effort, one can establish a similar result for cylinders $P_m \times C_n$.

Theorem 3.10. *For all $m, n \geq 0$ the kernel of rule σ^- on the $m \times n$ torus $C_m \times C_n$ has dimension $2 \cdot \gcd(m,n) - mn \bmod 2$. All tori are irreversible under rule σ^-.*

Proof. The argument is analogous to the proof of the last theorem; we will only state the recurrence relations. Again, we use the abbreviation $[m,n] := d(C_m \times C_n, \sigma^-)$.

$$[m,n] := \begin{cases} 2n & \text{if } m = 0 \vee (n = m \wedge m \text{ even}) \\ 2n-1 & \text{if } n = m \wedge m \text{ odd} \\ [n,m] & \text{if } n < m \\ [m, 2m-n] & \text{if } m < n < 2m \\ [m, n \bmod 2m] & \text{if } n \geq 2m. \end{cases} \tag{3.6}$$

The situation for cylinders of the form $C_m \times P_n$ is slightly more complicated due to the fact that $C_n \times P_n$ is isomorphic to $C_n \times P_m$ only in the trivial case $m = n$. The recurrence relations for cylinders must therefore reduce both arguments m and n separately.

Theorem 3.11. *For all $m, n \geq 0$ the kernel of rule σ^- on the $m \times n$ cylinder $C_m \times P_n$ has dimension*

$$d(C_m \times P_n, \sigma^-) = \begin{cases} \gcd(m, n+1) - 1 & \textit{if } 0 = o_2(m) = o_2(n+1) \\ \gcd(m, n+1) & \textit{if } o_2(m) < o_2(n+1) \textit{ or} \\ & 0 < o_2(m) = o_2(n+1) \\ 2\gcd(m, n+1) - 2 & \textit{otherwise.} \end{cases} \quad (3.7)$$

Hence the cylinder $C_m \times P_n$ is reversible under rule σ^- iff m and $n+1$ are relatively prime and either both m and $n+1$ are odd or the exponent of 2 in the prime decomposition of m is strictly larger than the exponent of 2 in the prime decomposition of $n+1$.

Proof. Again we will only state the recurrence relations. We use the abbreviation $[m, n] = d(C_m \times P_n, \sigma^-)$.

$$[m, n] := \begin{cases} 0 & \text{if } n = 0 \\ 2n & \text{if } m = 0 \\ m & \text{if } m = n+1 \wedge m \text{ even} \\ m - 1 & \text{if } m = n+1 \wedge m \text{odd} \\ m & \text{if } 2m = n+1 \\ [2n+2-m, n] & \text{if } n+1 < m < 2n+2 \\ [m \bmod 2n+2, n] & \text{if } 2n+2 \leq m \\ [m, n \bmod m] & \text{if } m \leq n \wedge m \text{ even} \\ [m, 2m-2-n] & \text{if } m \leq n \leq 2m-2 \wedge m \text{ odd} \\ [m, n \bmod 2m] & \text{if } n \geq 2m. \end{cases} \quad (3.8)$$

By way of comparison, the second-order Lindenmayer automaton on P_∞ using rule $\underline{\sigma}^+$ shows far more complicated behavior. The function $\gamma(P_m, \sigma^+)$ or $\gamma(C_m, \sigma^+)$ are highly irregular, as witnessed by table 2 which shows these values for $m \leq 40$. Note that the behavior of P_{40} is radically different from C_{40}; figure 5 shows a complete period of seed configurations 20, 19+20+21 on C_{40} and the first 120 generations obtained from seed configurations 1, 1+2 on P_{40} (recall that both C_{40} and P_{40} are assumed to have vertex set [40]). The complete period on P_{40} has length over two million. Also observe that $\gamma(P_4, \sigma^+) = 4$, we do not know whether any square other than $P_{4,4}$ has the property that it maximizes the dimension of the kernel of σ^+. Table 3 lists the co-rank of σ^+ for grids $P_m \times P_n$, $3 \leq m, n \leq 40$. We have been unable to find a representation for the co-rank of σ^+ even for squares $P_m \times P_m$.

To conclude, we will prove some results about the polynomials $\pi_n = \pi_n[\tau]$ in $F_2[\tau]$ (as opposed to specific quotients $\pi_n[\sigma_G]$) that can be used to derive general properties of σ-automata on product graphs. Geometrically, the results are quite obvious from figure 6 and its self-similarity properties. However, we will provide purely algebraic proofs. To obtain a somewhat more explicit description of π_n than the one given in (3.2), let $c_{n,j}$ in F_2 be the coefficient of term τ^j in π_n, i.e.,

$$\pi_n = \sum_{0 \leq j \leq n} c_{n,j} \tau^j.$$

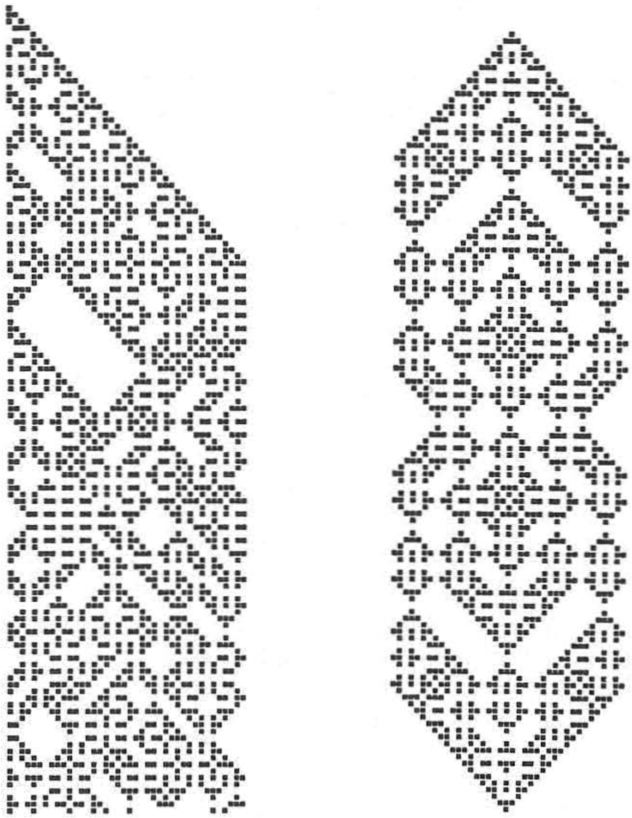


Figure 5: (a) The first 120 generations of an orbit on the $\underline{\sigma}^+$-automaton on the path of length 40. (b) The first period of an orbit on the $\underline{\sigma}^+$-automaton on the cycle of length 40.

m	P_m	C_m	m	P_m	C_m
1	2	5-	21	371-	1169
2	3	2	22	4093-	185-
3	11-	5	23	95	6140
4	9-	5-	24	2049-	47
5	23	14	25	251-	3074
6	17-	11	26	2043	125-
7	23-	8	27	71-	3065
8	27	11-	28	6553-	35-
9	59-	41	29	2039	9830
10	61-	29-	30	681-	1019
11	47	92	31	95-	1022
12	125-	23	32	4091	47-
13	35-	62	33	1019-	2045
14	339	17-	34	8189-	509-
15	47-	509	35	335	4094
16	509-	23-	36	7181-	167
17	167	254	37	2051-	3590
18	1025-	83	38	16379	1025-
19	119-	512	39	239-	8189
20	2339	59-	40	2097149-	119-

Table 2: The values of $\gamma(P_m, \sigma^+)$ and $\gamma(C_m, \sigma^+)$ for $m \leq 40$ (- indicates $\gamma < \gamma$).

The following recurrence relations hold:

$$c_{00} := 1$$

$$c_{10} := 0, c_{11} := 1$$

$$c_{k+2,0} := c_{k,0}$$

$$c_{k+2,j+1} := c_{k+1,j} + c_{k,j+1}.$$

The coefficients $c_{n,j}$ can again be generated by a one-dimensional second-order σ-automaton. In fact, the automaton this time is "one-way". The underlying graph is $G := \langle N, E_1, E_2 \rangle$ where $E_1 := \{(u,u) | u \geq 0\}$ and $E_2 := \{(u, u+1) | u \geq 0\}$. The two seed configurations are $Z_0 = 0$ and $Z_1 = 1$. Figure 6 shows the first 100 generations of configurations obtained in this way; note that the resulting fractal structure resembles the one obtained from P_∞ and σ^- after a rotation. Both structures have fractal dimension $\log_2 3$; the coefficients $c_{n,j}$ thus should not be expected to have any simple description. In any case, a straightforward induction yields the next proposition.

Proposition 3.12. *Let $0 \leq j \leq n$. The coefficient $c_{n,j}$ is zero whenever $n+j$ is odd. For $n+j$ even we have* $c_{n,j} = \binom{(n+j)/2}{j} \bmod 2$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1:		1			1			1			1			1			1			1			1			1			1			1			1			1		
2:	1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2	
3:		2			3			2			3			2			3			2			3			2			3			2			3			2		
4:				4					4					4					4					4					4					4					4	
5:	1	1	3		2		4	1	1		4		1	1	4		2		3	1	1		5		1	1	3		2		4	1	1		4		1	1	4	
6:								6									6									6									6					
7:		2			4			2			7			2			4			2			7			2			4			2			7			2		
8:	1		2		1	6	2		1		2		7		2		1		2	6	1		2		1		8		1		2		1	6	2		1		2	
9:		1		4	1			1	8		1			5			1		8	1			1	4		1			9			1		4	1			1	8	
10:																														10										
11:	1	2	3		4		7	2	1		6		1	2	8		4		3	2	1		10		1	2	3		4		8	2	1		6		1	2	7	
12:																																								
13:		1			1			7			1			1			13			1			1			7			1			1			13			1		
14:	1		2	4	1		2		5		2		1	4	2	8	1		6		1		2	4	1		2		5		2		9	4	2		1		6	
15:		2			4			2			8			2			4			2			15			2			4			2			8			2		
16:														8		8													8				8							
17:	1	1	3		2	6	4	1	1		4		13	1	4		2		3	7	1		5		1	1	15		2		4	1	1	6	4		1	1	4	
18:																																								
19:		2		4	3			2	8		3			6			3		16	2			3	4		2			11			2		4	3			2	16	
20:	1		2		1		2	6	1		2		1		2		7		2		1		2		1	6	2		1		2		1		8		1		2	
21:		1			1			1			1			1			1			1			1			1			1	10		1			1			1		
22:																																								
23:	1	2	3		5		7	2	1		10		1	2	15		5		3	2	1		16		1	2	3		5		16	2	1		10		1	2	7	
24:				4					4					4					4					4					4					4					4	
25:		1			1			1			1			1			1			1			1			1			1			1			1			1		
26:	1		2		1	6	2		1		2		7		2		1		2	6	1		2		1		8		1		2		1	6	2		1		2	
27:		2			3			8			3			2			15			2			3			8			3			2			27			2		
28:																																								
29:	1	1	3	4	2		4	1	9		4		1	5	4	8	2		11	1	1		5	4	1	1	3		10		4	1	17	4	4		1	1	12	
30:										10											10									20		10								
31:		2			4			2			8			2			4			2			16			2			4			2			8			2		
32:	1		2		1		2		1		2		1		2		1		2		1		2		1		2		1	10	2	20	1		2		1		2	
33:		1			1			1			1			9		8	1			1			1			1			17			1	16		1			1		
34:				4				6	4					4			6		4					4		6			4					4	6				4	
35:	1	2	3		4	6	7	2	1		6		13	2	8		4		3	8	1		10		1	2	27		4		8	2	1	6	6		1	2	7	
36:																																								
37:		1			1			1			1			1			1			1			1			1			1			1			1			1		
38:	1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2		1		2	
39:		2		4	4			2	8		7			6			4		16	2			7	4		2			12			2		4	7			2	32	
40:																																								
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

Table 3: Dimension of kernel of σ^+ for grids $P_m \times P_n$. Blanks indicate a 0.

In order to apply the last proposition, a simple method to determine the parity of binomial coefficients is needed. One useful criterion for the oddness of binomial coefficients can be obtained as follows. Let x be a natural number, $0 \leq x < 2^t$. The binary expansion of x may be construed as a bit-vector describing a subset S_x of $\{0, \ldots, t-1\}$. Note that for $0 \leq y \leq x : S_y \subseteq S_x$ iff $S_{x-y} \subseteq S_x$. The following proposition is proved in [7].

Proposition 3.13. *Let $0 \leq y \leq x$. Then $\begin{pmatrix} x \\ y \end{pmatrix}$ is odd iff $S_y \subseteq S_x$.*

For numbers n with simple binomial expansion, this allows in conjunction with proposition 3.12 to calculate π_n explicitly. The next two theorems are examples of this procedure.

Theorem 3.14. *Let $n = 2^\nu - 1$, $\nu \geq 1$. Then $\pi_n = \tau^n$. Hence $d(G \times [n]) =$ co-rank(σ_G^n). In particular, the σ-automaton on $G \times [n]$ is reversible iff the σ-automaton on G is reversible.*

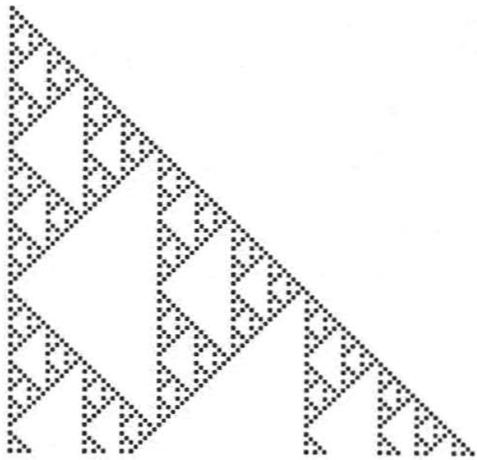


Figure 6: The coefficients of the polynomials π_n for $n \leq 100$. A box represents a 1, and a blank represents a 0 in F_2.

Proof. By our assumption $n = 2^\nu - 1$, thus n is odd and the binary expansion $[n]_2$ of n has the form 11..11. By proposition 3.12 and 3.13, $c_{n,j} = 1$ iff j is odd and $S_j \subseteq S_{(n+j)/2}$. Say, $j = 2k+1$, $0 \leq k \leq 2^{\nu-1} - 1$, and we get $c_{n,j} = 1$ iff $S_{2k+1} \subseteq S_{2^{\nu-1}+k}$. The latter condition clearly holds only for $k = 2^{\nu-1} - 1$. Thus $\pi_n = \tau^n$ and we are done. ∎

The following theorem is a stronger version of proposition 3.1.

Theorem 3.15. *For all* $n \geq 0$: $\pi_{2n+1} = \pi \circ \pi_n \circ \pi_n$.

Proof. As F_2 has characteristic two we have

$$\pi_n \circ \pi_n = \sum_{j \leq n} c_{n,j} \tau^{2j}.$$

Thus it suffices to show that $c_{2n+1,2j+1} = c_{n,j}$ for all $j = 0, \dots, n$ ($c_{2n+1,2j} = 0$ by proposition 3.12). Note that the binary expansion of $2n+1$ has the form $[2n+1]_2 = [n]_2 1$. Similarly, $[2j+1]_2 = [j]_2 1$. Therefore, $S_j \subseteq S_n$ iff $S_{2j+1} \subseteq S_{n+1}$ and we are done by propositions 3.12 and 3.13. ∎

According to theorem 3.6, for every m we have $d(P_{m,n}, \sigma^+) = m$ where $n := \gamma(P_m, \sigma^+)$. As mentioned above, we do not know how to compute $\gamma(P_m, \sigma^+)$ from m in any other than the brute force way. However, for special values of m, a simple description of $\gamma(P_m, \sigma^+)$ is available as expressed in the following lemma. Note that for these m $\gamma(P_m, \sigma^+)$ grows linearly in m.

Lemma 3.16. $m = 2^\nu - 1$ *and* $n = 3 \cdot 2^{\nu-1} - 1$, $\nu \geq 1$. *Then the kernel of* σ^+ *on* $P_{n,m}$ *has (the maximal) dimension* m: $d(P_{m,n}, \sigma^+) = m$.

Proof. Using propositions 3.12 and 3.13 one verifies that $\pi_n[\tau] = \tau^{2^{\nu-1}-1} + \tau^n$. Now $\sigma^+_{P_m}$ is an automorphisms of C_{P_m} and one can show that its order in the group of automorphisms is 2^ν (i.e., 2^ν is the least number $k \geq 1$ such that $(\sigma^+_{P_m})^k = \text{id}$). Therefore, we have $(\sigma^+_{P_m})^{2^\nu} = \text{id}$ and

$$\pi_n[\sigma^+_{P_m}] = (\sigma^+_{P_m})^{2^{\nu-1}-1} + (\sigma^+_{P_m})^n = 0.$$

Hence $d(P_{m,n}, \sigma^+) = \text{co-rank}(\pi_n[\sigma^+_{P_m}]) = m = \min(n, m)$ is maximal. ∎

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