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Exact Solutions for some Discrete Models of the Boltzmann Equation

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Abstract. For the simplest of the discrete models of the Boltzmann equations, the Broadwell model, exact solutions have been obtained by Cornille [15,16] in the form of bisolitons. In the present paper, we build exact solutions for more complex models.

1. Introduction

For the last twenty years, the study of discrete models of the Boltzmann equation has attracted the attention of many scientists. The first models, with six or eight velocities, were proposed in 1964 by J. Broadwell [1,2]. After Broadwell, R. Gatignol has written the general form of equations which represent the discrete models of the Boltzmann equation [3,4]. Those models are obtained by assuming that the molecules of a gas can have only a finite number of velocities, $\vec{u}_j$. With this assumption, the Boltzmann equation is replaced by a semi-linear hyperbolic system of partial differential equations. We denote the density of molecules with velocity $\vec{u}_j$ by $N_j(t,\vec{x})$ (t time, $\vec{x}$ position), and the discrete models of the Boltzmann equation are also written in the following form:

$$\frac{\partial N_j}{\partial t} + \vec{u}_j \cdot \vec{\nabla} N_j = \sum_{jkl} A_{jk}^{lm}(N_l N_m - N_j N_k)$$

$$(j = 1, 2, \ldots, p). \tag{1.1}$$

The coefficients A_{jk}^{lm}, transition probabilities, are constants, positive or zero; they depend on three of the indices j, k, l, m. The equations of R. Gatignol, (1.1), are the kinetic equations. Since 1974, the global existence of solutions of equation (1.1) has been proved for more and more complex models by Nishida and Mimura [5], Crandall and Tartar [6], Cabannes [7,8], Kawashima [9], Illner [10], and Tartar [11]; more recently, the bounded character of solutions has been proved by Beale [12,13] and Alvès [14].

For the simplest discrete model, the Broadwell model [1], exact non-trivial and physically acceptable solutions have been obtained by Cornille [15,16]; those solutions which depend on one space variable are rational functions of one or two exponential variables of the form $\exp(\rho t + \gamma x)$: solitons or bisolitons. In this paper, we obtain in a similar way exact solutions of the R. Gatignol equations for some models more complex than the Broadwell model.

2. Study of the general case

We look for solutions of equation (1.1), which have the following form:

$$N_j(t,x) = \alpha_j + 2\mathcal{R}\mathrm{e}\{a_j \tan(\lambda x + i\mu t)\}, \quad i = \sqrt{-1}. \tag{2.1}$$

The coefficients α_j and the two parameters λ and μ are real constants; the coefficients a_j are complex constants; $\mathcal{R}\mathrm{e}(z)$ denotes the real part of the complex number z, the conjugate of which is noted $\bar{z}$; the right-hand side of equation (2.1) represents bisolitons. The explicit form of the densities is also:

$$N_j(t,x) = \alpha_j + 2\,\frac{\beta_j \sin X - \gamma_j \,\mathrm{sh}\, T}{\cos X + \mathrm{ch}\, T} \tag{2.2}$$

where $X = 2\lambda x$, $T = 2\mu t$, $a_j = \beta_j + i\,\gamma_j$ (β_j and γ_j real). If we define $D = \cos X + \mathrm{ch}\, T$, we obtain:

$$\left.\begin{aligned} D^2\,\frac{\partial N_j}{\partial t} &= -4\mu\Big\{\beta_j \,\mathrm{sh}\, T \sin X + \gamma_j(1 + \mathrm{ch}\, T \cos X)\Big\} \\ D^2\,\frac{\partial N_j}{\partial t} &= 4\lambda\Big\{\beta_j(1 + \mathrm{ch}\, T \cos X) - \gamma_j \,\mathrm{sh}\, T \sin X\Big\} \end{aligned}\right. \tag{2.3}$$

$$\begin{aligned} D^2 N_l N_m &= (1 + \mathrm{ch}\, T \cos X)\alpha_l\alpha_m \\ &+ 2D \sin X(\alpha_l\beta_m + \alpha_m\beta_l) \\ &- 2D \,\mathrm{sh}\, T(\alpha_l\gamma_m + \alpha_m\gamma_l) \\ &+ \sin^2 X(4\beta_l\beta_m - \alpha_l\alpha_m) \\ &+ \mathrm{sh}^2 T(4\gamma_l\gamma_m + \alpha_l\alpha_m) \\ &- 4 \sin X \,\mathrm{sh}\, T(\beta_l\gamma_m + \beta_m\gamma_l) \end{aligned} \tag{2.4}$$

If we denote by p the number of discrete velocities $\vec{u}_j$ (with components u_j on the axis of abscissae x), the set of the functions $N_j(t,x)$ depends on $3p+2$ unknowns; the densities are periodic functions of the variable x.

In order that equation (1.1) is satisfied, it is sufficient to identify the terms in: $1 + \mathrm{ch}\, T \cos X$, $D \sin X$, $D \,\mathrm{sh}\, T$, $\sin^2 X$, $\mathrm{sh}^2 T$, $\sin X \,\mathrm{sh}\, T$.

We obtain also six relations:

$$4\lambda u_j\beta_j - 4\mu\ \gamma_j = \sum_{klm} A_{jk}^{lm}(\alpha_l\alpha_m - \alpha_j\alpha_k) \tag{2.5}$$

$$\sum_{klm} A_{jk}^{lm}(\alpha_l\beta_m + \alpha_m\beta_l - \alpha_j\beta_k - \alpha_k\beta_j) = 0 \tag{2.6}$$

$$\sum_{klm} A_{jk}^{lm}(\alpha_l\gamma_m + \alpha_m\gamma_l - \alpha_j\gamma_k - \alpha_k\gamma_j) = 0 \tag{2.7}$$

$$\sum_{klm} A_{jk}^{lm}(4\beta_l\beta_m - \alpha_l\alpha_m - 4\beta_j\beta_k + \alpha_j\alpha_k) = 0 \tag{2.8}$$

$$\sum_{klm} A_{jk}^{lm}(4\gamma_l\gamma_m + \alpha_l\alpha_m - 4\gamma_j\gamma_k - \alpha_j\alpha_k) = 0 \tag{2.9}$$

$$4\lambda u_j\gamma_j + 4\mu\beta_j = 4\sum_{klm} A_{jk}^{lm}(\gamma_l\beta_m + \gamma_m\beta_l - \gamma_j\ \beta_k - \gamma_k\beta_j) \tag{2.10}$$

The introduction of conservation equations, which are related to the summational invariants, allows to replace the system (1.1) by a simpler one [3,4]. To each summational invariant $\mathbf{V}^{(r)}$, vector of $\mathbf{R}^{(p)}$, with components $V_j^{(r)}$, corresponds a conservation equation:

$$\sum_{j=1}^{p} V_j^{(r)}\left(\frac{\partial N_j}{\partial t} + u_j\frac{\partial N_j}{\partial x}\right) = 0 \qquad (r = 1, 2, \ldots, q). \tag{2.11}$$

The number q of the conservation equations is equal to the dimension of the space of summational invariants $(1 \le q < p)$. Equation (2.11) will be satisfied if the two following relations are verified:

$$\sum_{j=1}^{p} V_j^{(r)}(\lambda u_j\beta_j - \mu\beta_j) = 0 \qquad (r = 1, 2, \ldots, q), \tag{2.12}$$

$$\sum_{j=1}^{p} V_j^{(r)}(\lambda u_j\gamma_j + \mu\beta_j) = 0 \qquad (r = 1, 2, \ldots, q). \tag{2.13}$$

Each conservation equation can replace one of the equations (1.1). There remain $6(p-q)+2q$ relations (2.5) through (2.10) and (2.12) through (2.13) to determine $3p+2$ constants. The number of relations is smaller or equal to the number of unknowns α_j, a_j, λ, and μ. One has $3p \le 4q+2$. To be physically acceptable, the densities $N_j(t, x)$ must be positive for all x, and for all $t \ge t_o$.

3. Examples of exact solutions

From a cube, one can build 14-velocity models: 6 velocities orthogonal to the faces, with moduli u and densities M_k; and 8 velocities parallel to the diagonals, with moduli v and densities N_j. Two of those models, and only two, allow mixed collisions. Mixed collisions are defined as collisions between molecules with density N_j and molecules with density M_k [17]. The first model has $u\sqrt{3} = v$, and the second model has $u\sqrt{3} = 2v$. For both models, one has $p = 14$ and $q = 5$. The functions, (2.1), depend on 44 real parameters which must satisfy 64 equations. Although the number of equations is greater than the number of unknowns, solutions exist and we will describe them.

If we assume that, for both sets of molecules, the densities of the molecules are equal for those velocities which have the same component on the axis of abscissae, then there exist only $p = 5$ different densities, and the number of conservation equations is equal to $q = 3$. In the case of the first model, the kinetic equations, equations (1.1), are the following [18]:

$$\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x} = \tfrac{\sqrt{6}}{2}\Big(N_2M_4 - N_1M_1\Big) \tag{3.1}$$

$$\frac{\partial N_2}{\partial t} + \frac{\partial N_2}{\partial x} = \tfrac{\sqrt{6}}{2}\Big(N_1M_1 - N_2M_4\Big) \tag{3.2}$$

$$\frac{\partial M_1}{\partial t} + \frac{\partial M_1}{\partial x} = 2\sqrt{6}\Big(N_2M_4 - N_1M_1\Big) + \frac{4}{3}\Big(M_2^2 - M_1M_4\Big) \tag{3.3}$$

$$\frac{\partial M_4}{\partial t} - \frac{\partial M_4}{\partial x} = 2\sqrt{6}\Big(N_1M_1 - N_2M_4\Big) + \frac{4}{3}\Big(M_2^2 - M_1M_4\Big) \tag{3.4}$$

$$\frac{\partial M_2}{\partial t} = + \frac{2}{3}\Big(M_1M_4 - M_2^2\Big) \tag{3.5}$$

The variables x and t are dimensionless. The conservation equations, deduced from equations (3.1) through (3.5), are

$$\frac{\partial N_2}{\partial t} + \frac{\partial N_2}{\partial x} = -\left(\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x}\right)$$

$$\frac{\partial M_1}{\partial t} + \frac{\partial M_1}{\partial x} = 4\left(\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x}\right) - 2\frac{\partial M_2}{\partial t}$$

$$\frac{\partial M_4}{\partial t} - \frac{\partial M_4}{\partial x} = -4\left(\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x}\right) - 2\frac{\partial M_2}{\partial t} \tag{3.6}$$

To present the results, we adopt the numbering of index j in formulae (2.1) according the order of equations (3.1) through (3.5), that means that we put $N_3 = M_1$, $N_4 = M_4$, and $N_5 = M_2$. If we change x to $-x$, the flow is reversed, and N_1 becomes $N_{2'}$, N_3 becomes N_4. Therefore, we can assume $\alpha_2 = \alpha_1$, $\beta_2 = -\beta_1$, $\gamma_2 = \gamma_1$, $\alpha_4 = \alpha_3$, $\beta_4 = -\beta_3$, $\gamma_4 = \gamma_3$ and $\beta_5 = 0$. We use next the relations (2.5) through (2.10) in equations (3.1) and (3.5) and the relations (2.12) and (2.13) in equation (3.6). We obtain the nine following homogeneous relations:

$$
\begin{aligned}
\lambda\beta_1 &+ \mu\lambda_1 = 0\\
\lambda\gamma_1 &- \mu\beta_1 = \sqrt{6}\,(\beta_1\gamma_3 + \beta_3\gamma_1)\\
\alpha_1\beta_3 &+ \alpha_3\beta_1 = 0\\
4\mu\gamma_5 &= \alpha_5^2 - \alpha_3^2\\
\alpha_3\gamma_3 &- \alpha_5\gamma_5 = 0\\
4\beta_3^2 &+ \alpha_3^2 - \alpha_5^2 = 0\\
4\gamma_3^2 &+ \alpha_3^2 - \alpha_5^2 - 4\gamma_5^2 = 0
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\mu\beta_3 &+ \lambda\gamma_3 = 4(\mu\beta_1 - \lambda\gamma_1) \qquad (3.8)\\
\mu\gamma_3 &- \lambda\beta_3 = 2\mu\gamma_5
\end{aligned}
$$

There are ten unknowns. The method of solution consists of eliminating α_j, β_j, and γ_j to obtain an algebraic equation, the unknown of which is the ratio $\sigma = \lambda/\mu$. We find two possibilities. First, for $\alpha_1 = \beta_1 = \gamma_1 = 0$, we obtain:

$$\sigma^4 + 6\sigma^2 - 3 = 0; \quad \text{that is,} \quad \sigma^2 = 2\sqrt{3} - 3. \tag{3.9}$$

$$\alpha_3 = \alpha(1-\sqrt{3}), \quad \beta_3 = \frac{\alpha\sigma}{2}, \quad \gamma_3 = -\frac{\alpha}{2}$$

$$\alpha_5 = \alpha, \quad \gamma_5 = -\frac{\alpha_3}{2}, \qquad \lambda = \sigma\mu = \sigma\alpha\,\frac{3-\sqrt{3}}{4} \tag{3.10}$$

That solution is the solution of Cornille [15]. If $\beta_1 \neq 0$, the elimination of α_1, β_1, γ_1 gives

$$
\begin{aligned}
(1+\sigma^2)\,\mu &+ \sqrt{6}\,(\gamma_3 - \sigma\beta_3) = 0\\
2\gamma_5 &+ \gamma_3 + \sigma\beta_3 = 0\\
3\mu\gamma_5 &= \alpha_5^2 - \alpha_3^2 = 4\beta_3^2\\
\alpha_3\gamma_3 &- \alpha_5\gamma_5 = 0 \text{ and } \gamma_5^2 = \gamma_3^2 - \beta_3^2
\end{aligned}
\tag{3.11}
$$

and we obtain

$$5(3-2\sqrt{6})\,\sigma^4 - 2(3+8\sqrt{6})\,\sigma^2 + 3\,(5-2\sqrt{6}) = 0, \tag{3.12}$$

σ^2 is again the positive root of equation (3.12), and after introduction of two constants τ and ω defined by formulae (3.13), we obtain the definitive results (3.14); we have in fact $\sigma = \pm 0.082$, $\omega = 1.128$ and $\tau = \pm 1.043$. The numerical values given in formulae (3.14) correspond to the case $\alpha > 0$ and $\tau > 0$; α is an arbitrary numerical constant (positive or negative).

$$\tau = \frac{(10-3\sqrt{6})\,\sigma^2 - 3(2-\sqrt{6})}{16\sigma}, \qquad 2\omega(1+\sigma\tau) = \sqrt{6} \tag{3.13}$$

$$\begin{aligned}
\lambda &= \omega\sigma\alpha = 0.092\ \alpha, \qquad \mu = \omega\alpha = 1.128\ \alpha \\
\alpha_1 &= \alpha_2\ \frac{\omega\tau(1+\sigma^2)}{4(\tau-\sigma)\sqrt{6}}\ \alpha = 0.126\ \alpha \\
a_1 &= -\bar{a}_2 = -\frac{\omega\tau(1-i\sigma)}{2\sqrt{6}}\alpha = \Big(-0.120 + 0.010\ i\Big)\alpha \\
\alpha_3 &= \alpha_4 = -\frac{(1+\sigma^2)}{2(1+\sigma\tau)}\ \alpha = -0.464\ \alpha \\
a_3 &= -\bar{a}_4 = \frac{1}{2}\Big(\frac{\sigma-\tau}{1+\sigma\tau} - i\Big)\alpha = -(0.443 + 0.500\ i)\alpha \\
\alpha_5 &= \alpha, \qquad a_5 = \frac{i\ \omega(1+\sigma^2)}{2\sqrt{6}} = 0.232\ i\ \alpha
\end{aligned} \tag{3.14}$$

To finish, we must show that the densities so determined are positive for all x and for $t \geq t_o$. We assume that T is positive; we have then $\cos X + \operatorname{ch} T > 0$, and it is sufficient to satisfy the conditions:

$$P_j = \alpha_j \cos X + 2\,\beta_j \sin X + \alpha_j \operatorname{ch} T - 2\,\gamma_j \operatorname{sh} T > 0. \tag{3.15}$$

For a fixed value of T, P_j varies between two extreme values, which are functions of T; we show that the smallest of those values is positive. In the case where the three constants ω, σ, and α are positive, we find that the conditions $P_j > 0$ are realized for all values of the index j, if we take $t_o \geq 0.22$. The variation of the density N_j with X is periodic and oscillates between the two extreme values:

$$N_j^m = \alpha_j - 2\,\frac{\gamma_j \operatorname{ch} T \pm \mid a_j \mid}{\operatorname{sh} T}. \tag{3.16}$$

When T increases, the minimum increases and the maximum decreases; the difference between the two values tends to zero, as T approaches infinitely.

4. Study of the second 14-velocity model

For the second 14-velocity model, the kinetic equations (3.1) through (3.5) must be replaced by the following:

$$\begin{aligned}
\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x} &= \sqrt{11}\ \Big\{N_2(M_2+M_4) - N_1(M_2+M_1)\Big\} \\
\frac{\partial N_2}{\partial t} - \frac{\partial N_2}{\partial x} &= \sqrt{11}\ \Big\{N_1(M_2+M_1) - N_2(M_2+M_4)\Big\} \\
\frac{\partial M_1}{\partial t} + 2\,\frac{\partial M_1}{\partial x} &= 4\ \Big\{M_2^2 - M_1M_4\Big\} + 4\sqrt{11}\ \Big\{N_2M_2 - N_1M_1\Big\} \\
\frac{\partial M_4}{\partial t} - 2\,\frac{\partial M_4}{\partial x} &= 4\ \Big\{M_2^2 - M_1M_4\Big\} + 4\sqrt{11}\ \Big\{N_1M_2 - N_2M_4)\Big\} \\
\frac{\partial M_2}{\partial t} &= 2\ \Big\{M_1M_4 - M_2^2\Big\} + \sqrt{11}\Big\{N_1M_1 + N_2M_4 - M_2(N_1+N_2)\Big\}
\end{aligned} \tag{4.1}$$

There are also three conservation equations:

$$\frac{\partial N_1}{\partial t} - \frac{\partial N_1}{\partial x} = -\Big(\frac{\partial N_2}{\partial t} + \frac{\partial N_2}{\partial x}\Big)$$

$$\frac{\partial}{\partial t}\Big(M_1 + M_4 + 4M_2\Big) + 2\frac{\partial}{\partial x}\Big(M_1 - M_4\Big) = 0$$

$$\frac{\partial}{\partial t}(M_1 - M_4 - 4N_1) + \frac{\partial}{\partial x}(2M_1 + 2M_4 + 4N_1) = 0 \tag{4.2}$$

Using the same notations and procedures used in the former section, we have again for σ^2 two possibilities. First, $\alpha_1 = \beta_1 = \gamma_1 = 0$ and $4\sigma^2 = 2\sqrt{3}-3$, which corresponds to the solution of Cornille [15,16] for the densities M_k. Second, σ^2 is a root of the following equation:

$$2184\sigma^6 + 4264\sigma^2 + 2306^2\sigma + 226 - \sqrt{11}\{696\sigma^6 + 1562\sigma^4 + 640\sigma^2 + 71\} = 0. \tag{4.3}$$

Equation (4.3), considered as a third-degree equation for the unknown σ^2, has two complex roots and a real root σ^2 = - 7.56583. As the real root is negative, σ is never real and no solution of the form (2.1) exists for equation (4.1), that is, for the second 14-velocity model.

5. Other examples

There exist several other models for which the kinetic equations (1.1) possess exact solutions of the form (2.1). We consider the four following equations.

$$\begin{aligned}
\frac{\partial N_1}{\partial t} + \frac{\partial N_1}{\partial x} &= k\Big(N_2N_4 - N_1N_3\Big)\\
\frac{\partial N_2}{\partial t} + v\frac{\partial N_2}{\partial x} &= \Big(N_1N_3 - N_2N_4\Big)\\
\frac{\partial N_3}{\partial t} - \frac{\partial N_3}{\partial x} &= k\Big(N_2N_4 - N_1N_3\Big)\\
\frac{\partial N_4}{\partial t} - v\frac{\partial N_4}{\partial x} &= \Big(N_1N_3 - N_2N_4\Big)
\end{aligned} \tag{5.1}$$

k and v are two positive constants. Five different discrete models are described by equation (5.1): two-dimensional regular models with

6 velocities	($0x$ on $\vec{u}_1$)	$k = 2,\ 2v = 1$
8 velocities	($0x$ on the bisector of $\vec{u}_1\ \vec{u}_2$)	$k = 1,\ v = \sqrt{2} - 1$

and three-dimensional regular models related to the dodecahedron [17]:

12 velocities	($0x$ on $\vec{u}_1$)	$k = 5,\ 5v = \sqrt{5}$
12 velocities	($0x$ on a diagnoal of the dodecahedron)	$k = 1,\ v = \sqrt{5}-2$
20 velocities	($0x$ orthogonal to a face of the dodecahedron)	$k = 1,\ v = \sqrt{5}-2$

Of course, in each case, the number of different densities is only four, because we consider the solution for which two densities are equal if they correspond to molecules with a velocity having the same component on the axis of the abscissae. For the models represented by equation (5.1), there are four kinetic equations ($p = 4$) and three conservation equations ($q = 3$)

$$\frac{\partial N_1}{\partial t}+\frac{\partial N_1}{\partial x}=\frac{\partial N_3}{\partial t}-\frac{\partial N_3}{\partial x}=-k\Big(\frac{\partial N_2}{\partial t}+v\frac{\partial N_2}{\partial x}\Big)=-k\Big(\frac{\partial N_4}{\partial t}-v\frac{\partial N_4}{\partial x}\Big) \tag{5.2}$$

We put as before $N_j = \alpha_j + 2\,Re\,\{a_j\ \tan\,(\lambda x + i\mu t)\}$ and $\alpha_3 = \alpha_1$, $\alpha_4 = \alpha_2$, $a_3 = -\bar{a}_1$, $a_4 = -\bar{a}_2$, and $\alpha_j = \beta_j + i\gamma_j$, $\lambda = \sigma\mu$. The algebric equations to satisfy are:

$$\begin{array}{ll} \beta_1+\sigma\gamma_1=0, & \beta_2+\sigma v\gamma_2=0, \\ (1+\sigma^2)\gamma_1+k(1+\sigma^2v^2)\gamma_2=0, & \\ 2(1+\sigma^2)\mu\gamma_1=k(\alpha_1^2-\alpha_2^2), & \alpha_2\gamma_2=\alpha_1\gamma_1, \\ (1+\sigma^2)\mu\gamma_1=k\{(1+\sigma^2v^2)\gamma_2^2-(1+\sigma^2)\gamma_1^2\}, & \\ (1-\sigma^2)\gamma_1^2=(1-\sigma^2v^2)\gamma_2^2). & \end{array} \tag{5.3}$$

From equation (5.3), we deduce the equation:

$$\frac{1-\sigma^2v^2}{1-\sigma^2}=k^2\Big\{\frac{1+\sigma^2v^2}{1+\sigma^2}\Big\}^2. \tag{5.4}$$

For all the five cases considered, this equation considered as an equation for σ^2 has only real roots, and only one positive root:

$$\begin{array}{ll} k=1,\ \ v=\sqrt{2}-1 & :\ \sigma^2=8.81256 \\ k=1,\ \ v=\sqrt{5}-2 & :\ \sigma^2=21.45355 \\ k=2,\ \ 2v=1 & :\ \sigma^2=0.59067 \\ k=5,\ \ 5v=\sqrt{5} & :\ \sigma^2=0.91439 \end{array}$$

When σ^2 is known, we obtain:

$$\begin{array}{lll} \gamma_2=\gamma, & \gamma_1=-k\gamma\frac{1+\sigma^2v^2}{1+\sigma^2} & \\ \beta_1=-\sigma\gamma_1, & \beta_2=-\sigma v\gamma_2 & \\ \alpha_1=2\varepsilon\gamma_2, & \alpha_2=2\varepsilon\gamma_2 & (\varepsilon=\pm1) \\ \mu=\frac{2\sigma^2v^2}{1+\sigma^2v^2}\ \frac{1-v^2}{1-\sigma^2}\gamma, & \text{and} & \lambda=\sigma\mu. \end{array} \tag{5.5}$$

The limits of densities when $t \to \pm\infty$ are

$$\lim_{t\to\infty} N_i(x,t) = \alpha_i - 2\gamma_i \text{sgn}\mu$$

$$\lim_{t\to-\infty} N_i(x,t) = \alpha_i + 2\gamma_i \text{sgn}\mu$$

In order to obtain $\alpha_i - 2\gamma_i\text{sgn}\mu$ for $i = 1$ and 2, it is necessary to choose $\mu < 0$, that means $\gamma(1-v^2) > 0$. Then we have

$$\lim_{t\to\infty} N_i(x,t) = 2\varepsilon(\gamma_1 + \gamma_2)$$

$$\lim_{t\to-\infty} N_1(x,t) = 2\varepsilon(\gamma_2 - \gamma_1)$$

$$\lim_{t\to-\infty} N_2(x,t) = 2\varepsilon(\gamma_1 - \gamma_2).$$

We choose ε in order that the product $\varepsilon(\gamma_1+\gamma_2)$ is positive. Also there exists a time, t_o, so that all the densities $N_i(x,t)$ are positive for $t \geq t_o$. It is possible to choose the initial time $t = t_1$ so that two of the densities are negative (the two others are then positive) and nevertheless the solution of the initial values problem for equation (5.1) exists for all values of $t \geq t_1$. In the classical theorems of global existence for the R. Gatignol equations, one assumes that the initial values are positive; but this assumption is not necessary, as it has been proved by Balabane for the Carleman equations [20].

6. Conclusions

With exact solutions of R. Gatignol equations, it is interesting to compute the functions H of Boltzmann:

$$H(t,x) = \sum_{j=1}^{5} N_j(t,x)\ log\ N_j(t,x) \tag{6.1}$$

$$\mathcal{H}(t) = \int_0^{\frac{\pi}{\lambda}} H(t,x)dx \tag{6.2}$$

The integral over one period $\mathcal{H}(t)$ is found to be a decreasing function of the time, but locally the function $H(t,x)$ is not always monotonic, in particular for $x = \pi/2\lambda$; the theorem-H of Boltzmann is, of course, a global theorem.

Beside the bisolitons (2.1) it is possible to find other families of exact solutions of equation (1.1); this has been done by Golse for the Broadwell model [19]. The solutions of Golse are self-similar solutions obtained by putting $tN_j(t,x) = \tilde{N}_j(\xi)$ with $\xi = x/t$. This transformation is also valid for the general case of equation (1.1), which are transformed in a system of pure differential equations for the functions $\tilde{N}_j(\xi)$. Unfortunately, in contrast with the bisolitons (2.1), the self-similar densities do not remain positive for all values of x.

An interesting remark can be made from the solutions we have built. All the authors who have proved theorems of global existence start with the assumption that the initial values of densities are positive. With the bisolitons, we have examples of global existence when the values of densities are not all positive.

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