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Eddy Viscosity and Diffusivity: Exact Formulas and Approximations

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Abstract. Exact asymptotic expressions for eddy diffusivity and eddy viscosity are obtained as the leading terms of infinite-series representations of integral equations which express the action of turbulence on an infinitesimal mean field. The series are transformed term by term from Eulerian to Lagrangian form. The latter is more suitable for constructing approximations to the exact asymptotic expressions. The analysis is prefaced by some qualitative remarks on possible improvements of eddy transport algorithms in turbulence computations.

1. Introduction

Eddy viscosity and eddy diffusivity have long been fruitful concepts in turbulence theory, and their use has made possible the computation of turbulent flows at Reynolds numbers too high for full numerical simulation. However, there is a fundamental logical flaw. Molecular viscosity is a valid concept when there is a strong separation of space and time scales between hydrodynamic modes and gas-kinetic collision processes. In high-Reynolds-number turbulence, on the other hand, there is typically a continuous range of significantly excited modes between the largest motions and those small motions which are represented by an eddy viscosity.

In the present paper, the lack of clean scale separation of modes is expressed by exact statistical equations in which the interaction between a (large-scale) mean field and a (small-scale) fluctuating field is nonlocal in space and time. In an asymptotic case of interaction between modes whose space and time scales are strongly separated, the exact formulas reduce to ones which are effectively local in space and time and which express what can be termed the distant-interaction eddy viscosity. Even in this asymptotic case, the exact eddy viscosity is not a simple expression, and it is not perfectly reproduced by any approximations that have been proposed.

Most of the present paper is devoted to a derivation of the exact expressions for asymptotic eddy viscosity and eddy diffusivity, the embedding of

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these expressions within infinite-series representations of the general nonlocal integral equations, and, particularly, the transformation of Eulerian formulas into Lagrangian ones. The Lagrangian representation is probably the one in which approximations to the exact eddy viscosity and eddy diffusivity can most successfully be carried out. It is hoped that both the Eulerian and Lagrangian exact formulas can be of use in analyzing approximations and interpreting various approaches to the construction of eddy transport coefficients.

This mathematical analysis is prefaced by a qualitative discussion of the more difficult question of improving eddy-viscosity and eddy-diffusivity algorithms actually used for subgrid-scale representation in computations of turbulent flows. The spatial and temporal nonlocalness exhibited in the mathematical analysis may here be of some practical significance. The relative success of very crude eddy-viscosity approximations in computations suggests that the dynamics of turbulence yields robust statistics, with feedback characteristics that somehow partly compensate for bad approximations. However, existing subgrid-scale approximations do not perform well in the computation of the point-to-point amplitude structure of a large-scale flow, as opposed to statistics. It may be that here the incorporation of nonlocal effects is essential. Nonlocality in time means that the subgrid modes exert reactive as well as resistive forces on the explicit modes, and this may be important in reproducing finite-amplitude instabilities and other properties of the explicit modes.

One consequence of the lack of clean separation of explicit and subgrid modes is that the latter exert fluctuating driving forces on the explicit modes which are conceptually distinct from eddy viscosity (or even negative eddy viscosity) [1]. Since the detailed structure of the subgrid modes is unknown in a flow computation, the fluctuating forces must be treated statistically, but the close coupling between the two classes of modes means that the relevant statistics are not purely random. The existence of fluctuating forces on the explicit modes implies that the explicit velocity field in a calculation is not simply replaceable by its statistical mean.

Some simple numerics yield a strong motivation for improvement of subgrid representations. If the smallest spatial scale treated explicitly in a high-Reynolds number flow increases by a factor c, the computational load of the explicit calculation decreases by a factor of perhaps c^4. The precise ratio depends on the method of computation used. If the calculation is Eulerian, and logarithmic factors associated with fast Fourier transforms are ignored, the power four used above arises from increase in grid mesh size in three dimensions together with increase in the smallest time step needed, the latter determined by the convection by the large-scale flow. If these crude estimates are relevant, an increase of minimum explicit scale by a factor 2 decreases the load by a factor 16, and an increase by a factor 4 decreases the load by a factor 256. This means that an improvement in subgrid representation that permits a shrinking of the explicit scale range by a factor 2 may be cost-effective, provided that it increases the computation size by less than a factor 16 over a cruder subgrid-scale representation.

Caveats should be stated at this point. First, a useful subgrid algorithm must be practical to program and implement. This implies, among other things, that it must have a reasonably broad application. Second, a new algorithm must in fact be an improvement. An analytical approximation which includes higher-order effects may actually make things worse rather than better, because the convergence properties of the relevant approximation sequences are subtle and dangerous.

It seems unlikely that adding correction terms to the local asymptotic formulas exhibited in the body of the present paper is a valid route to improving eddy-viscosity and eddy-diffusivity representations. And, of course, no eddy-viscosity representation, however good, takes account of the random forces exerted by the subgrid modes. Moreover, the statistical facts about the subgrid scales needed to evaluate even the lowest-order asymptotic formulas are unavailable in a practical flow calculation.

One alternative approach is to infer as much as possible about the behavior of the subgrid modes by extrapolation from the dynamics and statistics of the explicitly computed modes. The well-known Smagorinsky eddy-viscosity formula [2] can be viewed as a simple example of this approach. Here, the effective eddy viscosity is determined from the local rate-of-strain tensor of the explicit velocity field by appeal to Kolmogorov inertial-range scaling arguments. It may be worthwhile, however, to extract substantially more detailed information from the explicit velocity field in order to estimate the dynamical effects of the subgrid scales.

Suppose, for example, the flow calculation is sufficiently large that a substantial range of spatial scales is included in the explicit velocity field. It is then possible to analyze the explicit field to extract information about the mean transfer of energy between different scale sizes, reactive interactions, and, as well, the fluctuating forces exerted by explicit modes of small scale on those of larger scale. Such an analysis could be performed individually on each calculated flow field as the computation proceeds. However, it might be more economical to try to build up library tables of results from which these quantities could be rapidly estimated for a given computation using relatively few measured parameters. In either event, the effects of subgrid modes on the explicit field could then be estimated by assuming similarity with interscale dynamics within the explicit field. This would seem a less drastic assumption than adopting idealized inertial-range dynamics for the subgrid scales. Of course, the similarity analysis could be modified by taking into account crucial dynamical differences between explicit and subgrid modes—for example, increase of molecular viscosity effects with wavenumber.

2. Exact Eulerian analysis for eddy diffusivity

The equation of motion for a passive scalar advected by an incompressible velocity field may be manipulated to yield statistical equations which are exact and which display an eddy-diffusivity term acting on scalar modes having very large space and time scales. Let the scalar field $\phi(\mathbf{x},t)$ obey

$$L(\kappa)\phi(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\phi(\mathbf{x},t)=0 \tag{2.1}$$

where

$$L(\kappa)=\frac{\partial}{\partial t}-\kappa\nabla^2 \tag{2.2}$$

and

$$\nabla\cdot\mathbf{u}(\mathbf{x},t)=0. \tag{2.3}$$

Write

$$\phi(\mathbf{x},t)=\phi^{av}(\mathbf{x},t)+\phi'(\mathbf{x},t) \tag{2.4}$$

where

$$\phi^{av}(\mathbf{x},t)=\langle\phi(\mathbf{x},t)\rangle \tag{2.5}$$

and $\langle\ \rangle$ denotes ensemble average. Assume

$$\langle\mathbf{u}(\mathbf{x},t)\rangle=0. \tag{2.6}$$

If, instead, $\mathbf{u}(\mathbf{x},t)$ has a nonzero mean, extra terms appear in the following analysis.

Equations (2.1) through (2.6) yield the following equations for the mean and fluctuating scalar fields:

$$L(\kappa)\phi^{av}(\mathbf{x},t)=Q(\mathbf{x},t) \tag{2.7}$$

$$[L(\kappa)+\mathbf{u}(\mathbf{x},t)\cdot\nabla)]\phi'(\mathbf{x},t)=-\mathbf{u}(\mathbf{x},t)\cdot\nabla\phi^{av}(\mathbf{x},t)-Q(\mathbf{x},t) \tag{2.8}$$

where

$$Q(\mathbf{x},t)=\langle\mathbf{u}(\mathbf{x},t)\cdot\nabla\phi'(\mathbf{x},t)\rangle. \tag{2.9}$$

Define the unaveraged Green's function $g(\mathbf{x},t;\mathbf{y},t)$ by

$$[L(\kappa)+\mathbf{u}(\mathbf{x},t)\cdot\nabla]g(\mathbf{x},t;\mathbf{y},s)=0 \qquad (t\geq s), \tag{2.10}$$

$$g(\mathbf{x},s;\mathbf{y},s)=\delta(\mathbf{x}-\mathbf{y}). \tag{2.11}$$

Assume

$$\phi(\mathbf{x},t)=0. \tag{2.12}$$

Then, equations (2.7) through (2.11) yield

$$\phi'(\mathbf{x},t)=-\int_0^t ds\int d\mathbf{y}\, g(\mathbf{x},t;\mathbf{y},s)[\mathbf{u}(\mathbf{y},s)\cdot\nabla_y\phi^{av}(\mathbf{y},s)+Q(\mathbf{y},s)] \tag{2.13}$$

and, therefore,

$$\begin{aligned} Q(\mathbf{x},t) &= \frac{\partial}{\partial x_i}\int_0^t ds \int dy\ \lambda_i(\mathbf{x},t;\mathbf{y},s)Q(\mathbf{y},s) \\ &+ \frac{\partial}{\partial x_i}\int_0^t ds \int dy\ \mu_{ij}(\mathbf{x},t;\mathbf{y},s)\frac{\partial \phi^{av}(\mathbf{y},s)}{\partial y_j}. \end{aligned} \tag{2.14}$$

where equation (2.3) is used and

$$\lambda_i(\mathbf{x},t;\mathbf{y},s) = \langle u_i(\mathbf{x},t)g(\mathbf{x},t;\mathbf{y},s)\rangle, \tag{2.15}$$

$$\mu_{ij}(\mathbf{x},t;\mathbf{y},s) = \langle u_i(\mathbf{x},t)g(\mathbf{x},t;\mathbf{y},s)u_j(\mathbf{y},s)\rangle. \tag{2.16}$$

No approximation has been made so far. Suppose that $\phi^{av}(\mathbf{x},t)$ has space and time scales very long compared to those that are significant in $\mathbf{u}(\mathbf{x},t)$. Then, equations (2.7) and (2.14) yield an equation of motion for $\phi^{av}(\mathbf{x},t)$ in which the coefficient functions are averages whose correlation scales are characteristic of the velocity field. Now let the slowly varying field $\phi^{av}(\mathbf{y},s)$ in (2.14) be expanded in a Taylor series about $(\mathbf{x},s)$. Assume that this series has at least a finite radius of convergence in

$$\xi_i = y_i - x_i$$

or that the series is asymptotic in some suitable sense about $\xi_i = 0$. Equation (2.14) may be expanded in the form

$$Q(\mathbf{x},t) = Y(\mathbf{x},t) + Z(\mathbf{x},t) \tag{2.17}$$

where

$$Y(\mathbf{x},t) = \frac{\partial}{\partial x_i}\int_0^t ds[\lambda_i(\mathbf{x},t;s)Q(\mathbf{x},t) + \lambda_{ij}(\mathbf{x},t;s)\ \frac{\partial Q(\mathbf{x},s)}{\partial x_j} + \ldots], \tag{2.18}$$

$$Z(\mathbf{x},t) = \frac{\partial}{\partial x_i}\int_0^t ds\left[\mu_{ij}(\mathbf{x},t;s)\ \frac{\partial \phi(\mathbf{x},s)}{\partial x_j} + \mu_{ijm}(\mathbf{x},t;s)\ \frac{\partial^2 \phi^{av}(\mathbf{x},s)}{\partial x_j \partial x_m} + \ldots\right], \tag{2.19}$$

and

$$\lambda_i(\mathbf{x},t;s) = \int dy \langle u_i(\mathbf{x},t)g(\mathbf{x},t;\mathbf{y},s)\rangle, \tag{2.20}$$

$$\lambda_{ij}(\mathbf{x},t;s) = \int dy \langle u_i(\mathbf{x},t)g(\mathbf{x},t;\mathbf{y},s)\xi_j\rangle, \tag{2.21}$$

etc.

$$\mu_{ij}(\mathbf{x},t;s) = \int dy \langle u_i(\mathbf{x},t)g(\mathbf{x},t;\mathbf{y},s)u_j(\mathbf{y},s)\rangle, \tag{2.22}$$

$$\mu_{ijm}(\mathbf{x},t;s) = \int d\mathbf{y} \langle u_i(\mathbf{x},t) g(\mathbf{x},t;\mathbf{y},s) u_j(\mathbf{y},s) \xi_m \rangle, \tag{2.23}$$

etc.

Equations (2.17) through (2.19) still provide a formally exact representation of $Q(\mathbf{x},t)$, implicitly in powers of a ratio ℓ_u/ℓ_ϕ, where ℓ_u and ℓ_ϕ are characteristic spatial scales of the velocity field and the mean scalar field ϕ^{av}, respectively. An equivalent of (2.22) was first derived by Corrsin [3].

$Q(\mathbf{x},t)$ may now be expressed in powers of ℓ_u/ℓ_ϕ by an iteration solution of equation (2.17). The leading term in this expansion is the term in $\mu_{ij}(\mathbf{x},t;s)$. If the s-variation of $\phi^{av}(\mathbf{x},s)$ is neglected in comparison to that of $\mu_{ij}(\mathbf{x},t;s)$, there results the equation of motion for $\phi^{av}(\mathbf{x},t)$:

$$L(\kappa)\phi^{av}(\mathbf{x},t) = \frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x},t)\,\frac{\partial \phi^{av}(\mathbf{x},t)}{\partial x_j}\right] \tag{2.24}$$

where

$$\kappa_{ij}(\mathbf{x},t) = \int_0^t \mu_{ij}(\mathbf{x},t;s)ds. \tag{2.25}$$

The tensor $\kappa_{ij}(\mathbf{x},t)$ is the exact asymptotic eddy diffusivity in the sense that (2.24) is asymptotically exact if the spatial and temporal variation of $\phi^{av}(\mathbf{x},t)$ is infinitely slow compared to that of $\mathbf{u}(\mathbf{x},t)$. No closure approximation has been made. It is assumed, via equation (2.12), that the mean scalar field is switched on at $t = 0$ (or, equivalently, that the velocity field is switched on at $t = 0$). Statistical homogeneity or stationarity of the velocity field is not assumed, but if the velocity field is statistically homogeneous and isotropic, the exact asymptotic scalar eddy diffusivity (in three dimensions) is

$$\kappa_{eddy}(t) = \frac{1}{3}\kappa_{ii}(\mathbf{x},t). \tag{2.26}$$

3. Lagrangian transformation

There are two principal reasons for transforming the Eulerian formulae of section 2 into Lagrangian expressions. First, the results reproduce Taylor's 1921 formula [4] for eddy diffusivity when the molecular diffusivity κ vanishes, and thereby provide an exact generalization of Taylor's formula to the case of nonzero κ. Second, the Lagrangian expressions are physically more appropriate because they eliminate confusing (and canceling) convection effects in the amplitude factors from which the right-hand sides of (2.20) through (2.23) are constructed. Thereby, these expressions become more suitable for statistical approximation. This is discussed further in section 5.

The Lagrangian transformation of equations (2.18) and (2.19) (with equations (2.20) through (2.23) inserted), has the effect of changing the spacetime integrals on the right-hand sides into integrals backward in time from the present instant t, along a space-filling family of fluid-element trajectories. This can be done by introducing the generalized velocity field

$\mathbf{u}(\mathbf{x},t|s)$ and scalar field $\phi(\mathbf{x},t|s)$ defined as follows: $\mathbf{u}(\mathbf{x},t|s)$ is the velocity field measured at time s in that fluid element whose trajectory passes through the spacetime point (x,t), with a corresponding meaning for $\phi(\mathbf{x},t|s)$ [5]. The Eulerian fields are then

$$u_i(\mathbf{x},t) = u_i(\mathbf{x},t|t), \quad \phi(\mathbf{x},t) = \phi(\mathbf{x},t|t), \tag{3.1}$$

while the usual Lagrangian velocity field is

$$u_i^\ell(\mathbf{x},t) = u_i(\mathbf{x},0|t). \tag{3.2}$$

A generalized Green's function may also be defined: $g(\mathbf{x},t|s;\mathbf{x}',t'|s')$ is the probability density that scalar concentration found at time s' in the fluid element whose trajectory passes through $(\mathbf{x}',t')$ appears at time s in the fluid element whose trajectory passes through $(\mathbf{x},t)$. If $\kappa = 0$ (no molecular diffusivity),

$$g(\mathbf{x},t|s;x',t|s') = \delta(\mathbf{x}-\mathbf{x}') \quad \text{(all } s \text{ and } s'\text{)}. \tag{3.3}$$

That is, the scalar concentration in a fluid element is independent of the time of measurement. Equations of motion for the generalized fields and the generalized Green's function have been given elsewhere [5].

The transformation to Lagrangian representation is effected by expressing the integration in equations (2.20) through (2.23) in terms of the new vector variable $\mathbf{z}(\mathbf{y},t|s)$, defined in harmony with the preceding definitions as the position at time s of the fluid element whose trajectory passes through $(\mathbf{y},t)$. The transformation is volume preserving, in consequence of (1.3) so that $d\mathbf{z} = d\mathbf{y}$. It follows from the definitions of the generalized functions that

$$u[\mathbf{z}(\mathbf{y},t|s),s] \equiv \mathbf{u}(\mathbf{y},t|s), \text{ etc.} \tag{3.4}$$

Then, equations (2.20) through (2.23) may be rewritten as

$$\lambda_i(\mathbf{x},t;s) = \int d\mathbf{y}\langle u_i(\mathbf{x},t)g(\mathbf{x},t|t;\mathbf{y},t|s)\rangle, \tag{3.5}$$

$$\lambda_{ij}(\mathbf{x},t;s) = \int d\mathbf{y}\langle u_i(\mathbf{x},t)g(\mathbf{x},t|t;\mathbf{y},t|s)\xi_j(\mathbf{y},t|s)\rangle, \tag{3.6}$$

$$\mu_{ij}(\mathbf{x},t;s) = \int d\mathbf{y}\langle u_i(\mathbf{x},t)g(\mathbf{x},t|t;\mathbf{y},t|s)u_j(\mathbf{y},t|s)\rangle, \tag{3.7}$$

$$\mu_{ijm}(\mathbf{x},t;s) = \int d\mathbf{y}\langle u_i(\mathbf{x},t)g(\mathbf{x},t|t;\mathbf{y},t|s)u_j(\mathbf{y},t|s)\xi_m(\mathbf{y},t|s)\rangle, \tag{3.8}$$

where

$$\xi_i(\mathbf{y},t|s) = z_i(\mathbf{y},t|s) - x_i. \tag{3.9}$$

If $\kappa = 0$, it follows from (3.3) and (2.6) that (3.5) through (3.8) reduce to

$$\lambda_i(\mathbf{x},t;s) = 0, \tag{3.10}$$

$$\lambda_{ij}(\mathbf{x},t;s) = \langle u_i(\mathbf{x},t)\xi_j(\mathbf{x},t|s)\rangle, \tag{3.11}$$

$$\mu_{ij}(\mathbf{x},t;s) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t|s)\rangle, \tag{3.12}$$

$$\mu_{ijm}(\mathbf{x},t;s) = \langle u_i(\mathbf{x},t)u_j(\mathbf{x},t|s)\xi_m(\mathbf{x},t|s)\rangle. \tag{3.13}$$

Equation (3.12) is equivalent to Taylor's 1921 formula for eddy diffusivity, the difference being that the Lagrangian velocity in (3.12) is referred to positions of fluid elements at time t, while that in Taylor's formula is referred to positions at the initial instant. The entire expansion of which equations (3.10) through (3.13) give the leading terms is equivalent to an infinite-series equation of motion for $\phi^{av}(\mathbf{x},t)$ of Moffatt's Lagrangian form [6-8]. A difference is that the present expansion retains nonlocality in time, while in Moffatt's form only present values of the mean field appear. A further difference is that the reversion of power series which underlies Moffatt's form is not needed here. Moffatt's form of expansion gives an elegant expression of turbulent diffusion of weak magnetic fields [6–8]. However, serious complications arise if the same kind of expansion is attempted for eddy-viscosity effects. Although the vorticity field obeys the same formal equation of motion as the weak magnetic field, the intrinsic statistical dependence of velocity on vorticity plays a crucial role.

It should be noted that the case $\kappa = 0$ can be made to include the case of nonzero κ by the artifice of representing molecular diffusivity as due to a very rapid and small-scale component of the velocity field. Nevertheless, it is of practical interest to have the explicit formulae (3.5) through (3.8) for nonzero κ. Moreover, it can be useful to make only a partial transformation to Lagrangian coordinates, so as to lump small-scale, rapid modes of the turbulent velocity field with the molecular diffusivity. This can be done by using a filtered velocity field to define the Lagrangian transformation. Equations (3.5) through (3.8) retain the same forms, but the meaning of the generalized fields is altered. If $\mathbf{v}(\mathbf{x},t)$ is the filtered field obtained by passing the Eulerian field $\mathbf{u}(\mathbf{x},t)$ through a low-pass wavenumber filter or other appropriate filter, then the generalized fields are related to the Eulerian fields by [5]:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\right]u_i(\mathbf{x},t|s) = 0, \qquad u_i(\mathbf{x},s|s) = u_i(\mathbf{x},s), \tag{3.14}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\right]\phi(\mathbf{x},t|s) = 0, \qquad \phi(\mathbf{x},s|s) = \phi(\mathbf{x},s). \tag{3.15}$$

The vector field $\mathbf{z}(\mathbf{x},t|s)$ now represents trajectories of particles carried by the field $\mathbf{v}(\mathbf{x},t)$. Only if $\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t)$ and κ vanishes do equations (3.5) through (3.8) reduce to (3.10) through (3.13). In either event, (3.5) through (3.8) remain exact equations.

4. Exact analysis for eddy viscosity

The Navier-Stokes (N-S) equation leads to expansions like those developed in sections 2 and 3, and associated expressions may be constructed for the asymptotic eddy viscosity acting on slowly varying modes. However, there are important differences arising from the nonlinearity of the equations of motion and from the vector character of the velocity field. The nonlinearity implies that an asymptotic eddy viscosity, independent of the slow field it acts upon, is well-defined only if the slow field has infinitesimal amplitude. The vector nature of the field adds qualitatively new features to the expansions.

Let the N-S equation for an incompressible velocity field $\mathbf{u}(\mathbf{x},t)$ in a cyclic box or infinite domain be written as

$$L(\nu)u_i(\mathbf{x},t) = -\frac{1}{2}P_{ijm}(\nabla)[u_j(\mathbf{x},t)u_m(\mathbf{x},t)], \tag{4.1}$$

where L is defined by equation (2.2), ν is kinematic viscosity,

$$P_{ijm}(\nabla) = \nabla_m P_{ij}(\nabla) + \nabla_j P_{im}(\nabla), \tag{4.2}$$

$$P_{ij}(\nabla) = \delta_{ij} - \nabla^{-2}\nabla_i\nabla_j, \quad \nabla_i \equiv \frac{\partial}{\partial x_i}. \tag{4.3}$$

Here, $P_{ijm}(\nabla)$ expresses elimination of the pressure field via (2.3). $P_{ij}(\nabla)$ is a projection operator which suppresses the longitudinal part of the vector function on which it operates. Let $u_i^{av}(\mathbf{x},t)$ be an infinitesimal mean field switched on as a perturbation at time $t = 0$. Assume that the unperturbed field $u_i(\mathbf{x},t)$ has zero mean for all t, and let $u_i'(\mathbf{x},t)$ be the perturbation induced in the fluctuating field. Then, equation (4.1) yields the following equations for the evolution of $\mathbf{u}^{av}$ and $\mathbf{u}'$:

$$L(\nu)u_i^{av}(\mathbf{x},t) = Q_i(\mathbf{x},t), \tag{4.4}$$

$$Q_i(\mathbf{x},t) = -P_{ijm}(\nabla)\langle u_j(\mathbf{x},t)u_m'(\mathbf{x},t)\rangle, \tag{4.5}$$

$$\begin{aligned}[\delta_{im}L(\nu) \; &+ \; P_{ijm}(\nabla)u_j(\mathbf{x},t)]u_m'(\mathbf{x},t) \\ &= \; -P_{ijm}(\nabla)[u_j(\mathbf{x},t)u_m^{av}(\mathbf{x},t)] - Q_i(\mathbf{x},t).\end{aligned} \tag{4.6}$$

As in the scalar case, $u_i'(\mathbf{x},t)$ can be expressed as a spacetime integral over a Green's function. The latter is now a solenoidal tensor defined by

$$[\delta_{im}L(\nu) + P_{ijm}(\nabla)u_j(\mathbf{x},t)]g_{mn}(\mathbf{x},t;\mathbf{y},s) = 0, \tag{4.7}$$

$$g_{mn}(\mathbf{x},s;\mathbf{y},s) = P_{mn}(\nabla)\delta(\mathbf{x}-\mathbf{y}). \tag{4.8}$$

If the integral expression for $u_i'(\mathbf{x},t)$ is substituted into (4.5), the result may be written in a form analogous to (2.14):

$$\begin{aligned} Q_i(\mathbf{x},t) &= \nabla_j \int_0^t ds \int d\mathbf{y}[\lambda_{ijn}(\mathbf{x},t;\mathbf{y},s)Q_n(\mathbf{y},s) \\ &+ \mu_{ijan}(\mathbf{x},t;\mathbf{y},s)\frac{\partial u_n^{av}(\mathbf{y},s)}{\partial y_a} + \alpha_{ijn}(\mathbf{x},t;\mathbf{y},s)u_n^{av}(\mathbf{y},s)] \quad (4.9) \end{aligned}$$

where

$$\begin{aligned} \lambda_{ijn}(\mathbf{x},t;\mathbf{y},s) &= P_{im}(\nabla)\langle[u_j(\mathbf{x},t)g_{mn}(\mathbf{x},t;\mathbf{y},s) \\ &+ u_m(\mathbf{x},t)g_{jn}(\mathbf{x},t;\mathbf{y},s)]\rangle, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \mu_{ijan}(\mathbf{x},t;\mathbf{y},s) &= P_{im}(\nabla)\langle[u_j(\mathbf{x},t)g_{mr}(\mathbf{x},t;\mathbf{y},s) \\ &+ u_m(\mathbf{x},t)g_{jr}(\mathbf{x},t;\mathbf{y},s)]P_{rn}(\nabla_y)u_a(\mathbf{y},s)\rangle, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \alpha_{ijn}(\mathbf{x},t;\mathbf{y},s) &= P_{im}(\nabla)\langle[u_j(\mathbf{x},t)g_{mr}(\mathbf{x},t;\mathbf{y},s) \\ &+ u_m(\mathbf{x},t)g_{jr}(\mathbf{x},t;\mathbf{y},s)]P_{rb}(\nabla_y)\partial u_b(\mathbf{y},s)\partial y_n\rangle. \end{aligned} \quad (4.12)$$

In these equations, $P_{im}(\nabla_y)$ operates on functions of $\mathbf{y}$. Both the symmetry of $P_{ijm}(\nabla)$ and the property (2.3) are used in obtaining the right-hand sides. There are some differences from the scalar case. First, $\mu_{ijan}(\mathbf{x},t;\mathbf{y},s)$ and $\alpha_{ijn}(\mathbf{x},t;\mathbf{y},s)$ are operators; $P_{rn}(\nabla_y)$ and $P_{rb}(\nabla_y)$ operate on everything to their right. Second, the term in α_{ijn} has no counterpart in the scalar case. It involves $u_n^{av}(\mathbf{y},s)$ itself rather than its spatial derivative. This term is analogous to the α-effect term in the equation for a weak magnetic field diffused by turbulence [6].

As in the scalar case, $\mathbf{u}^{av}(\mathbf{y},s)$ can be expanded in power series about the point $(\mathbf{x},s)$. If the leading α_{ijn} term is nonzero, it is the leading term in the entire expansion, and as a consequence, in contrast to the scalar case, the λ_{ijn} term can contribute to the exact asymptotic eddy viscosity.

The power-series expansions are formally straightforward. If the leading α_{ijn} term vanishes, as it does in reflection-invariant homogeneous turbulence, the exact asymptotic form taken by (4.9) in limit of infinitely slowly varying $\mathbf{u}^{av}$ is

$$Q_i(\mathbf{x},t) = \nabla_j \nu_{ijan}(\mathbf{x},t)\frac{\partial u_n^{av}(\mathbf{x},t)}{\partial x_a} \quad (4.13)$$

where

$$\nu_{ijan}(\mathbf{x},t) = \int_0^t ds \int d\mathbf{y}[\mu_{ijan}(\mathbf{x},t;\mathbf{y},s) + \alpha_{ijn}(\mathbf{x},t;\mathbf{y},s)\xi_a]. \quad (4.14)$$

In equation (4.14), μ_{ijan} and $\alpha_{ijn}\xi_a$ reduce to functions; the P operators no longer act on $\mathbf{u}^{av}$. If the turbulence is statistically homogeneous, isotropic, and nonhelical, the exact asymptotic equation of motion for $\mathbf{u}^{av}$ reduces to

$$L(\nu)\mathbf{u}^{av}(\mathbf{x},t) = \nu_{eddy}(t)\nabla^2\mathbf{u}^{av}(\mathbf{x},t), \quad (4.15)$$

with

$$\nu_{eddy} = \frac{1}{9} P_{in}(\nabla)\nu_{ijjn}(\mathbf{x},t), \tag{4.16}$$

in three dimensions.

It should be noted that when $\nu_{ijan}(\mathbf{x},t)$ for homogeneous turbulence is transformed into the wavevector domain, the ξ_a factor in (4.14) becomes a derivative with respect to wavevector. Such derivatives do not appear in the wavevector representation of the asymptotic eddy diffusivity.

The Lagrangian transformation of the eddy viscosity can be carried out as in the scalar case, but care must be taken to correctly handle the derivatives with respect to $\mathbf{y}$; derivatives with respect to $\mathbf{z}(\mathbf{y},t|s)$ are not equivalent to derivatives with respect to $\mathbf{y}$. If the Eulerian fields

$$w_{rna}(\mathbf{y},s) = P_{rn}(\nabla_y)u_a(\mathbf{y},s), \tag{4.17}$$

$$\chi_{rna}(\mathbf{y},s) = P_{rb}(\nabla_y)\left[\xi_a \frac{\partial u_b(\mathbf{y},s)}{\partial y_n}\right] \tag{4.18}$$

are defined, the generalized fields $w_{rna}(\mathbf{y},t|s)$ and $\chi_{rna}(\mathbf{y},t|s)$ may be defined by an equation like (3.14). Then, (4.14) can be transformed to

$$\begin{aligned} \nu_{ijan}(\mathbf{x},t) &= P_{im}(\nabla)\int_0^t \int d\mathbf{y}\langle[u_j(\mathbf{x},t)g_{mr}(\mathbf{x},t|t;\mathbf{y},t|s) \\ &+ u_m(\mathbf{x},t)g_{jr}(\mathbf{x},t|t;\mathbf{y},t|s)]\,[w_{rna}(\mathbf{y},t|s) \\ &+ \chi_{rna}(\mathbf{y},t|s)]\rangle. \end{aligned} \tag{4.19}$$

The equations of motion for $g_{mr}(\mathbf{x},t|t;\mathbf{y},t|s)$ have been described elsewhere [5]. If the power-series expansion of $\mathbf{u}^{av}(\mathbf{y},s)$ of $(\mathbf{x},s)$ is carried to higher terms, the Lagrangian transformation can be carried out term by term in the entire resulting series for $Q_i(\mathbf{x},t)$.

5. Breaking the averages

The expressions for asymptotic eddy diffusivity and eddy viscosity derived in sections 2 through 4 are exact under the assumptions made. However, they involve rather complicated statistical averages. It is tempting to try to approximate the expressions by factoring them into products of simpler averages. Most of the closure approximations that have been proposed for eddy diffusivity and eddy viscosity involve such factoring, performed according to a variety of rationales.

Consider the approximation to the eddy diffusivity obtained by factoring the Eulerian expression (2.22) and inserting the result into (2.25):

$$\kappa_{ij}(\mathbf{x},t) = \int_0^t ds \int d\mathbf{y}[\langle u_i(\mathbf{x},t)u_j(\mathbf{y},s)\rangle\langle g(\mathbf{x},t;\mathbf{y},s)\rangle]. \tag{5.1}$$

What can be said about the validity of this approximation? First it may be noted that (5.1) alternatively can be obtained as a consequence of the direct-interaction approximation (DIA) for the turbulent diffusion of a passive scalar [9], and it has been proposed independently in the form of a natural approximation for the Lagrangian velocity covariance [3,10]. The DIA also yields equations which determine the evolution of the covariance and mean Green's function that appear in (5.1). Numerical integrations indicate that both (5.1) and the DIA values are good qualitative and quantitative approximations in isotropic turbulence, provided that the wavenumber spectrum of the Eulerian velocity field is concentrated about its center of gravity [11].

If instead the spectrum is diffuse in wavenumber, (5.1) introduces qualitative errors because of convection effects. (This is independent of whether the DIA is used to evaluate the right side.) To see this, suppose that the Eulerian velocity spectrum consists of a strong low-wavenumber part and a high-wavenumber part, the two widely separated in wavenumber. Convection of the high-wavenumber field by the low-wavenumber field will induce rapid decorrelation of the latter in time, and this makes both averages on the right side fall off rapidly. However, the product average, in the original exact expression (2.22), does not show this effect because the convection effects in the three factors are correlated.

The spurious convection effects in (5.1) do not arise if the Lagrangian average (3.7) is factored to give the approximation

$$\mu_{ij}(\mathbf{x},t;s) = \int d\mathbf{y}[\langle u_i(\mathbf{x},t)u_j(\mathbf{y},t|s)\rangle\langle g(\mathbf{x},t|t,\mathbf{y},t|s)\rangle]. \tag{5.2}$$

In the case $\kappa = 0$, the g function is statistically sharp with the value (3.3), and (5.2) is identical with the exact expression (3.12). If κ is nonzero, $g(\mathbf{x},t|t;\mathbf{y},t|s)$ fluctuates because of distortion of fluid elements by the flow. The distortion makes the scalar gradient in a given fluid element fluctuate and hence makes the molecular diffusion fluctuate. However, it is plausible that this fluctuation is much more weakly correlated with fluctuations in $u_j(\mathbf{x},t|s)$ than the convection effects which afflict the factoring of the Eulerian expression (2.22). Thus, it is plausible that (5.2) remains a good qualitative and quantitative approximation even for nonzero κ.

A more drastic approximation is to simply replace the g function in (3.7) by the value it would have in the absence of the turbulence:

$$g^0(\mathbf{x},t|t;\mathbf{y},t|s) = [4\pi\kappa(t-s)]^{-3/2}\exp\left(\frac{-|\mathbf{x}-\mathbf{y}|^2}{[4\kappa(t-s)]}\right). \tag{5.3}$$

This approximation ignores completely the effects of straining of fluid elements on molecular diffusion. It should, nevertheless, have a greater domain of validity than the Eulerian factoring (5.1).

Approximations to the exact asymptotic eddy viscosity also may be constructed by factoring the Eulerian and Lagrangian expressions. As in the scalar case, the Eulerian factoring introduces spurious convection effects when the velocity spectrum encompasses a wide range of wavenumbers. These effects do not arise in the factoring of (4.19), which leads to the approximation

$$\begin{aligned}\nu_{ijan}(\mathbf{x},t) &= P_{im}(\nabla)\int_0^t \int d\mathbf{y}[W_{jrna}(\mathbf{x},t;\mathbf{y},t|s)G_{mr}(\mathbf{x},t|t;\mathbf{y},t|s) \\ &+ W_{mrna}(\mathbf{x},t;\mathbf{y},t|s)G_{jr}(\mathbf{x},t;\mathbf{y},t|s)]. \end{aligned} \tag{5.4}$$

Here,

$$W_{jrna}(\mathbf{x},t;\mathbf{y},t|s) = \langle u_j(\mathbf{x},t)[w_{rna}(\mathbf{y},t|s) + \chi_{rna}(\mathbf{y},t|s)]\rangle \tag{5.5}$$

contains the Lagrangian velocity covariance, modified by the solenoidal projection operator, while

$$G_{mr}(\mathbf{x},t;\mathbf{y},t|s) = \langle g_{mr}(\mathbf{x},t;\mathbf{y},t|s)\rangle \tag{5.6}$$

is the mean Lagrangian Green's tensor.

In contrast to the scalar case, the inviscid Lagrangian Green's tensor is not simply the solenoidal projection of a δ-function; it exhibits effects of straining and pressure fluctuations. Consequently, replacing G_{mr} by its purely viscous value may not to be a valid further approximation in high-Reynolds-number turbulence. Perhaps more justified is the approximation of W_{jrna} by a simpler Lagrangian tensor:

$$\begin{aligned} W_{jrna}(\mathbf{x},t;\mathbf{y},t) &= P_{rn}(\nabla_y)U_{ja}(\mathbf{x},t;\mathbf{y},t|s) \\ &+ P_{rb}(\nabla_y)\langle u_j(\mathbf{x},t)\xi_a(\mathbf{y},t|s)\frac{\partial u_b(\mathbf{y},t|s)}{\partial y_n}\rangle \end{aligned} \tag{5.7}$$

where

$$U_{ja}(\mathbf{x},t;\mathbf{y},t|s) = \langle u_j(\mathbf{x},t)u_a(\mathbf{y},t|s)\rangle \tag{5.8}$$

is the Lagrangian velocity covariance.

Equations (5.2), (5.4), and (5.7) alternatively are obtained as consequences of the Lagrangian-history DIA [5], which provides approximations for the Lagrangian functions G, U_{ja}, and G_{mr} in terms of Eulerian quantities. It would be interesting to see how good an approximation (5.2) and (5.4) provide (with or without equation (5.7)) if exact values of G, G_{mr}, W_{jrna}, and U_{ja} are used.

The exact formulas of section 4 assume that the large-scale mean field has infinitesimal amplitude, but in typical applications, the modes of large spatial scale have most of the kinetic energy. It is therefore important to discuss the errors associated with finite mean-field excitation, both for the exact asymptotic formulas and for the approximate factorings presented in the present section. Modes of large spatial scale exert two (interacting) kinds of effect on modes of small spatial scale: convection and distortion. The former is proportional to the large-scale velocity and the latter to the large-scale rate-of-strain tensor. If the turbulence is statistically homogeneous, convection effects do not alter the energy transfer between the mean field and the small-scale turbulence and therefore do not alter the eddy viscosity. This is expressed formally by the invariance of the exact asymptotic expressions (4.10) through (4.12), (4.14), (4.16), and (4.19) to Galilean transformations. The exact eddy-diffusivity expressions (2.22), (2.25), and (3.12) also are invariant under Galilean transformation when there is statistical homogeneity. Distortion by the large-scale mean field does affect the energy transfer. Thus, the effective eddy viscosity exerted on a finite-strength large-scale field differs from the asymptotic value if the rates of strain associated with the large-scale field are comparable to those intrinsically associated with the small-scale turbulence.

Galilean invariance extends also to the factoring approximations (5.1), (5.2), and (5.4). In the case of the Eulerian factoring (5.1) and its counterpart for eddy viscosity, both the covariance and Green's function factors on the right-hand side change under a statistically sharp Galilean transformation, but the changes exactly compensate when there is statistical homogeneity.

The convection effects of large-scale *fluctuating* fields are associated with behavior under random Galilean transformation: the addition to the total velocity field of an $\mathbf{x}$-independent velocity which is randomly different in each realization [5]. Both the Eulerian and Lagrangian exact asymptotic eddy-diffusivity and eddy-viscosity formulas are invariant under random Galilean transformation, but the factored approximations behave differently. The Lagrangian factorings (5.2) and (5.4) are invariant because both covariance and Green's function factors are invariant. However, these factors both change under transformation in the Eulerian case (5.1) and the corresponding factoring for eddy-viscosity. Because of the breaking of the averages, the changes do not compensate, as they did for statistically sharp transformation, and the resulting eddy diffusivity and eddy viscosity show spurious change under the transformation (see the discussion following equation (5.1)).

Eddy viscosity traditionally is positive, corresponding to energy flow from the large-scale field to the small-scale turbulence. However, various negative-viscosity phenomena, in which the direction of energy flow is reversed, have been described [12–20]. It should be noted here that the negative viscosity effect in two-dimensional turbulence [12-15], which can occur with isotropic statistics, and the corresponding effect in three dimensions [15-20], which depends essentially on anisotropy of the small-scale statistics, are both contained in the exact asymptotic eddy expression (4.14). Moreover, they survive in the factoring approximation (5.4) and under the further approximation of using Lagrangian-history DIA to evaluate the covariance and Green's function.

The recently described "anisotropic kinetic alpha (AKA) effect" [20] is a reverse-energy-flow phenomenon that involves the α_{ijn} term in (4.9). There are several interesting features. First, the AKA effect results in growth of large-scale helical waves of a particular sign, and therefore, the small-scale field defines a screw sense. Nevertheless, the small-scale field has no helicity. Second, the maintenance of the AKA effect in steady state requires that non-Galilean-invariant, small-scale, zero-mean forcing terms be added to the Navier-Stokes equation. In the presence of such forcing, an additional term appears in (4.9) which involves the functional derivative of the Green's tensor with respect to the mean velocity field.

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