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Candidates for the Game of Life in Three Dimensions*

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Abstract. The game "Life" is defined in a strict sense and three candidates for three-dimensional versions are presented. One of these versions can be structured to contain an infinite number of parallel two-dimensional universes, each of which allows for the evolution of Conway life objects. Various oscillators are described, and a few interesting collisions between translating oscillators ("gliders") and other objects are mentioned.

1. Introduction–Conway's Game of Life

Most readers are probably familiar with John Conway's two-dimensional cellular automaton known as the "Game of Life" [3,4]. The game is "played" by zero players on an arbitrarily large grid of square cells, where each cell is either "alive" or "dead". Essentially, the game works as follows. Start at generation one with some pattern of living cells (squares on the grid that are filled in). To obtain the next generation, apply the following transition rules concurrently to each cell, C, on the grid, whether filled in or not. Rule One: If C is living and if it touches two or three living cells, it remains alive for the next generation; otherwise, C dies (i.e., erase the filled-in square for next generation). Rule Two: If C is not living and if it touches exactly three living cells, C becomes alive (i.e., fill C in for next generation).

Readers familiar with the game may recall that with appropriate starting patterns, we can obtain a host of stable and oscillating shapes, which Conway and others have given such whimsical names as "beehive", "blinker", "clock", "pulsar", etc. Several oscillators translate across the grid with successive generations; such oscillators are traditionally called *gliders*, a term which we shall use throughout this paper.

1.1 The rules of Life

We can formalize the rules for Life as follows. Define *environment* E as the number of living neighbors required to prevent a currently living cell

*A preliminary report on some of this work appeared in [1,2].

from expiring, with $E_l \leq E \leq E_u$. *Fertility* F is the number of neighbors required to create a new living cell, $F_l \leq F \leq F_u$. Define the *transition rule* R as the 4-tuple $(E_l E_u F_l F_u)$. For Conway's Life, $R = (2333)$.

Naturally, we can construct other rules to apply to a two-dimensional grid. For example, $R = (3434)$ yields a game known as "3–4" Life, which exhibits a variety of oscillators which are totally different from those resulting from $R = (2333)$. Unfortunately, it is easy to produce starting patterns in 3–4 Life that rapidly expand forever. True, we can force (2333) to produce forms that grow without limit, but the intriguing feature of (2333) is that this can only be done with carefully constructed configurations.

We could also enlarge the neighborhood, or introduce "aging", where a living cell dies after a certain number of generations. Work has also been done using a hexagonal instead of a square grid [5]. Conway's rule is much less complex than most of the hexagonal transition rules—and therein lies its beauty. Our goal here is to describe similar elegant rules in three dimensions which yield a large, interesting variety of stable shapes and oscillators, support one or more gliders, and when applied to any initial and relatively haphazard configuration of cells, will ultimately stabilize. Hence, we restrict our use of the name "Life" to only those rules that are, as Dewdney puts it, "worthy of the name" [2]. The following definition formalizes this restriction.

Definition 1. *A rule $E_l E_u F_l F_u$ defines a "Game of Life" if and only if both of the following are true.*

1. A glider must exist and must occur "naturally" if we apply $E_l E_u F_l F_u$ repeatedly to primordial soup configurations.
2. All primordial soup configurations, when subjected to $E_l E_u F_l F_u$, must exhibit bounded growth.

(Here we define *primordial soup* as any finite mass of arbitrarily dense randomly dispersed living cells.)

We have not specified in our definition just how many "soup experiments" to perform before we conclude that a glider does not exist; this question is best answered by considering the implication of definition 1—if a glider does not condense out of some haphazard arrangement of cells, then there is little hope of creating one by bombarding some configuration with a (man-made) glider. Thus, the "rarer" a glider is, the less likely that interesting configurations (e.g. a "glider gun"—a manufactured device which produces an endless supply of gliders) may exist.

2. Finding a rule for three-dimensional Life

In three dimensions, a cell can have from 0 to 26 living neighbors; hence, we may construct a huge variety of rules of the form described above. Specifically, we can have

$$(E_l, E_u) = (1,1), (1,2), (1,3), \ldots (1,26); (2,2), (2,3), \ldots \text{etc.}$$

for a total of $(26+25+\ldots 1) = 351$. A similar number of values for (F_l, F_u) gives a total of 123,201 possible rules. Fortunately, if we are looking for a rule that behaves in a manner similar to Conway's rule, we can restrict our scope considerably, since most of the possible combinations yield forms that expand rapidly and indefinitely or quickly shrink and disappear. The following theorems are of assistance.

Theorem 2. *Any rule $E_l E_u F_l F_u$ with $F_l \geq 10$ cannot support a glider.*

This is easily seen when one observes that a non-living cell adjacent to a plane of living neighboring cells can have at most nine neighbors—an upper bound for a cell adjacent to any plane grouping of cells. Thus, any formation under rule $(X\,Y\,10\,Z)$ will ultimately either shrink and disappear, or will form a convex blob whose outer surface of living cells may remain in turmoil, but will never translate across the universe (see figure 1).

Theorem 3. *Any rule $E_l E_u F_l F_u$ with $F_l \leq 4$ leads to unlimited growth.*

To prove theorem 3, simply start with a cluster of four neighboring cells arranged in a square (see figure 2 for a more exotic example).

After testing several possibilities, it becomes obvious that rules dealing with from four to seven neighbors have the most potential. Starting configurations that are operated upon by rules (5767), (5777), (5566), (5755), (4656), (4655), (6767), (4567), (6766), etc., seem to either vanish quickly, leaving little or no residue, or grow indefinitely. Furthermore, none of these rules seem to support a glider. Rule (5655) offers several interesting small oscillators (see figure 3), but its residue is rather sparse and an exhaustive search has revealed no glider. Rule (5877), as well as other rules whose environment range exceeds 3, leaves too much nameless debris and does not seem to yield particularly interesting configurations. Rule (4666) offers several interesting oscillators, but leaves much more residue than (4555) and does not appear to support a glider. The same can be said about (4566), (3455), and (3566). One should observe at this point that we can easily create rules which leave stable non-oscillating patterns just by utilizing a small fertility range and a large environment range. Furthermore, one can find an infinite supply of distinct oscillators with arbitrarily long periods simply by constructing dense "random" blobs and applying rules with $F_l > 10$; for example, the period of the oscillating blob in figure 1 under rule (10 21 10 21) exceeds 100.

2.1 The best rules

Of all the rules investigated, only $R = (4555)$ and $R = (5766)$ satisfy definition 1. (These games can be denoted "Life 4555" and "Life 5766".)

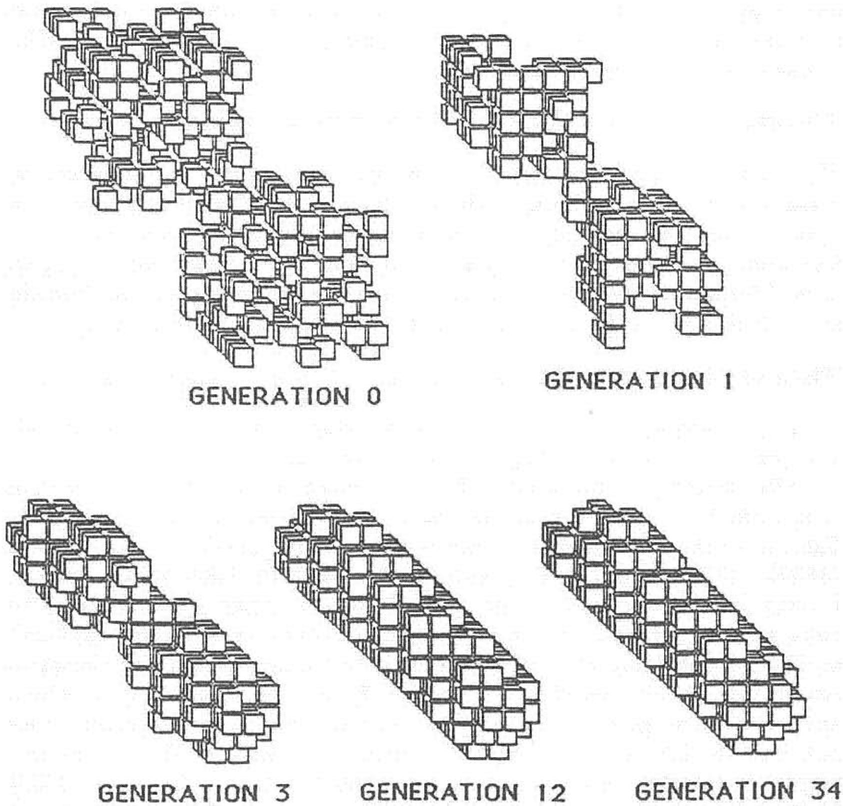

Figure 1: The rule (10 21 10 21) frequently leads to objects that look similar to the one shown. This object oscillates with a period in excess of 100. By starting with large initial objects, we can create oscillators with periods as long as we wish.

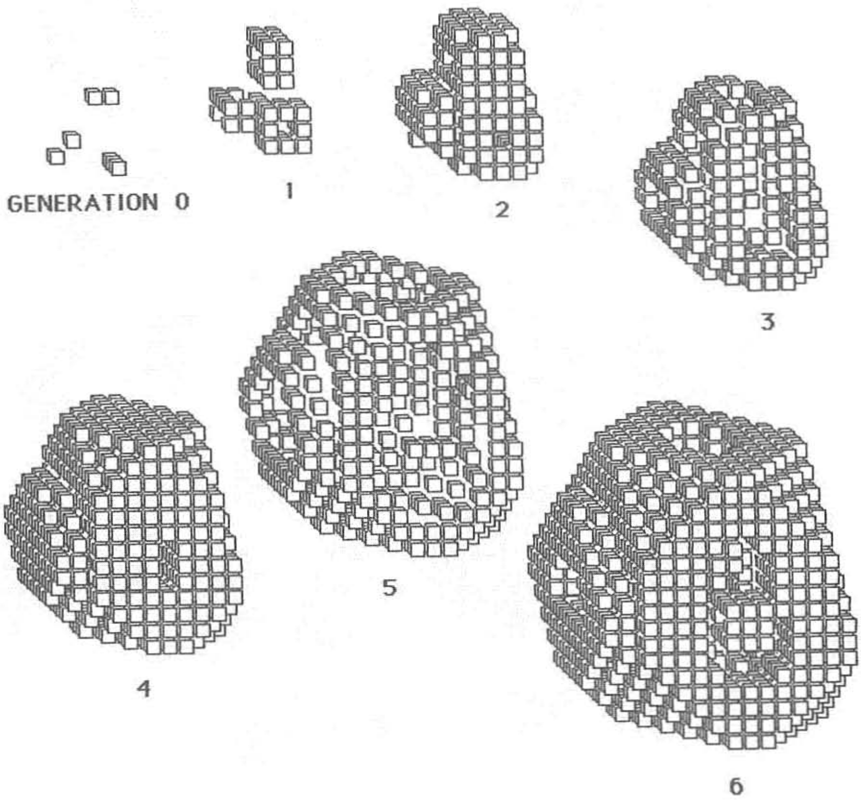

Figure 2: The rule (4526) leads to immediate unbounded growth. Here, the starting pattern was six centrally placed cells.

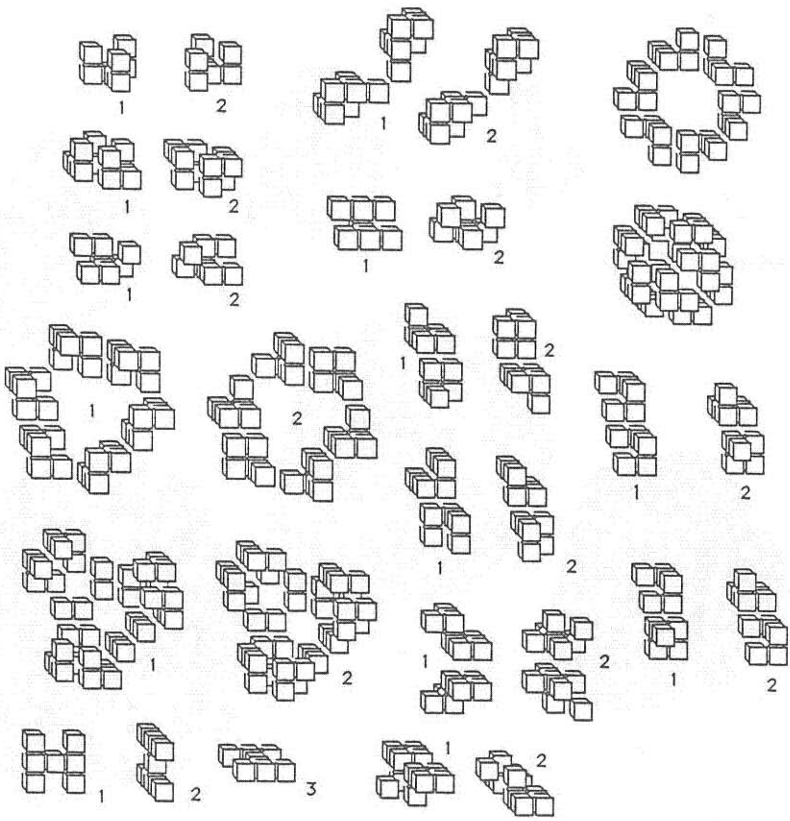

Figure 3: Some of the oscillators for the rule (5655). With initial random soup, (5655) produces very little residue. An exhaustive search has revealed that this rule supports no naturally occurring glider.

Rule (5766) leads to stable and oscillating forms (see figure 4) that are similar in many ways to Conway's (2333), as discussed below. One of the characteristics of Life 5766 is that the three environment states allow for a large number of small stable asymmetric objects. For example, if we confine our scope to the stable forms that can be contained within a $4 \times 4 \times 4$ cube, there are well over 100 varieties. (Figure 5 depicts just a few of the many stable shapes that can be created by removing from one to four cells from a 24-element stable object.)

Rule (4555), although somewhat less prolific than (5766), may ultimately be a more interesting rule. For one thing, (4555) requires more time to "settle down" than (5766); hence, there is more of a possibility for interesting intermediate reactions. Moreover, it is formed simply by adding 2 to Conway's rule, $R = (2333)$. Perhaps the most fascinating feature of (4555) is that there exist an abundance of small stable and oscillating forms that usually exhibit symmetry of some sort. This rule will be discussed in more detail in section 2.3.

2.2 A comparison between Conway's Life and three-dimensional Life

Before proceeding further, we should examine the relationship between the above three-dimensional Life rules and Conway's two-dimensional rule, $R = (2333)$. Define a *Conway object* as any configuration of cells, stable or not, that exists at some point during Conway's game. Cells in Conway objects will have coordinates $(x_i, y_i, 0)$; that is, the object lies in the $Z = 0$ plane. We shall further employ the following definitions.

Definition 4. *An* expansion *of a Conway object is formed in three dimensions by making copies of all living cells $(x_i, y_i, 0)$ into the adjacent Z plane, i.e. $(x_i, y_i, 1)$.*

Hence, the expansion has twice as many living cells as the original Conway object. It may or may not behave in an interesting manner when subjected to one of the three-dimensional Life rules: $R = (5766)$ or $R = (4555)$.

Definition 5. *A* projection *of a three-dimensional Life object into two dimensions exists if and only if both of the following are true.*

1. *All of the living cells (x_i, y_i, z_i) lie in two adjacent planes. For the sake of discussion, let these planes be $Z = 0$ and $Z = 1$.*
2. *The pair of cells $(x_i, y_i, 0)$ and $(x_i, y_i, 1)$ are either both alive or both dead.*

Definition 6. *An* analog *of a Conway object in three dimensions is an expansion which, when subjected to the appropriate three-dimensional Life rule, yields after each and every generation a projection identical to the original Conway object for the same generation under the two-dimensional rule, $R = (2333)$.*

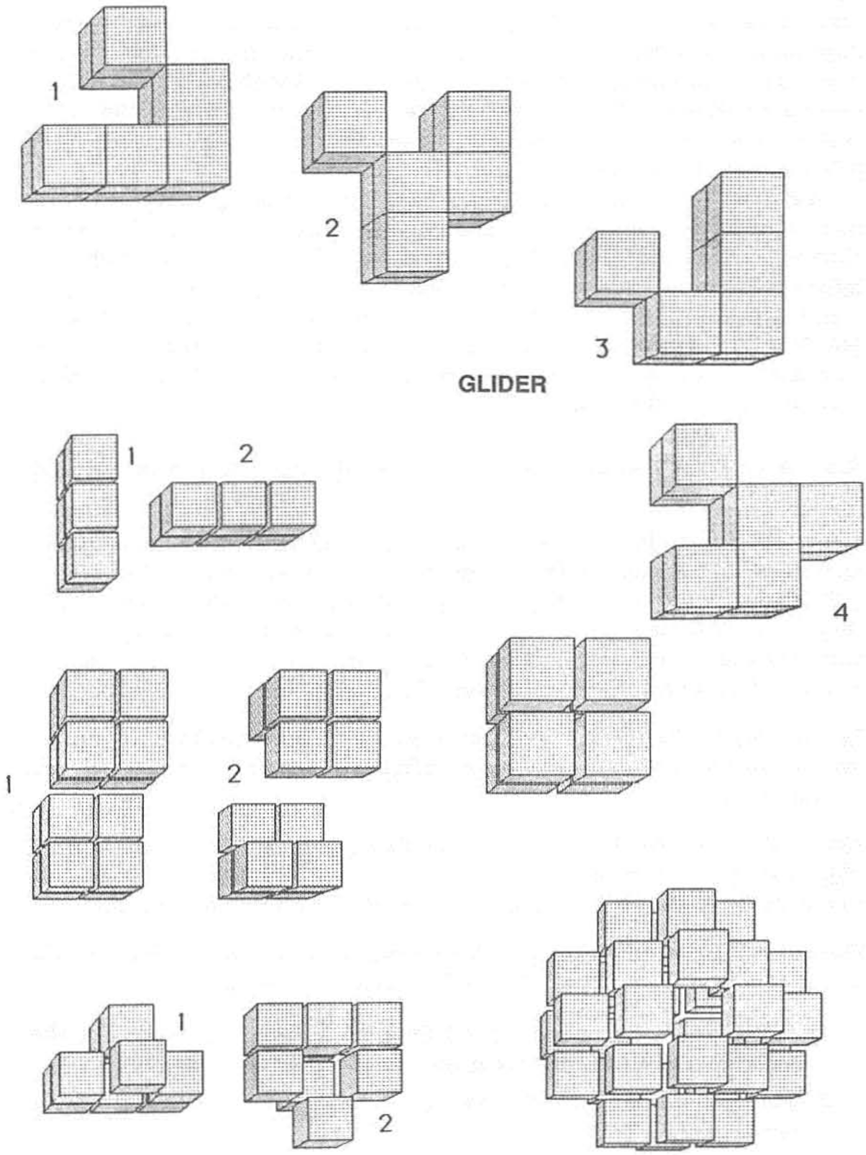

Figure 4: A few of the many small stable forms under Life 5766.

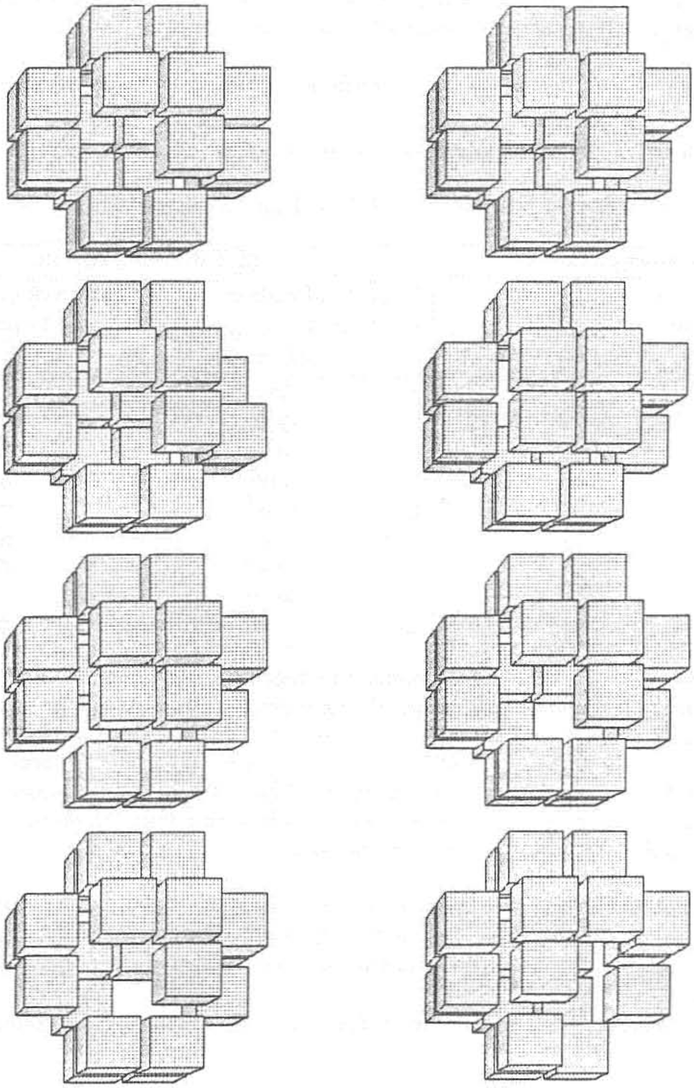

Figure 5: Here are just a few of the many stable shapes under Life 5766 that can be created by removing one or more cells from a 24-element symmetric stable object. There are too many such forms to illustrate.

The above definitions facilitate statement of the following theorem.

Theorem 7. *A Conway object has an analog under the three-dimensional Life rule $R = (5766)$ if and only if the Conway object has both the following characteristics at every (subsequent) generation:*

1. *A non-living cell in the neighborhood of the object cannot have six living neighbors.*
2. *A living cell cannot have five neighbors.*

The proof is obtained by examining tables 1 and 2.

Conway Object		(5 7 6 6) Expansion			
number of neighbors, N when cell at $(x_i, y_i, 0)$ is alive		N, state of cells at $(x_i, y_i, 0)$ and $(x_i, y_i, 1)$		N, state of cells at $(x_i, y_i, -1)$ and $(x_i, y_i, 2)$	
N	Next state	N	Next state	N	Next state
0	dead	1	dead	1	dead
1	dead	3	dead	2	dead
2	alive	5	alive	3	dead
3	alive	7	alive	4	dead
4	dead	9	dead	5	dead
5	dead	11	dead	6	alive
6	dead	13	dead	7	dead
7	dead	15	dead	8	dead
8	dead	17	dead	9	dead

Table 1: Comparison of the number of neighbors and the status for Conway cells and Life 5766 cells. For example, if a Conway cell is alive and has four neighbors (line 4 in the table), then next generation it will die, as will the pair of cells in the Life 5766 expansion. When a cell in the Conway object has five neighbors, the next generation of 5766 expansion will have new live cells in plants adjacent to the expansion, thereby destroying the analog.

Notice that the behavior of the expansion in $Z = 0$ and $Z = 1$ under $R = (5766)$ is identical to Conway's Life. A deviation only occurs in the $Z = -1$ and $Z = 2$ planes; these deviations are the restrictions imposed by theorem 1.

Upon further examination of tables 1 and 2, we obtain the following corollary.

Corollary 8. *The three-dimensional Life rule $R = (5766)$ yields behavior that is more analogous to Conway's Life than any other three-dimensional rule that we may construct as $R = (E_l, E_u, F_l, F_u)$.*

The implications of theorem 7 are startling. If one examines all the small stable and oscillating Conway forms, one notices that a great many

Conway Object		(5 7 6 6) Expansion			
number of neighbors, N, when cell at $(x_i, y_i, 0)$ is dead		N, state of cells at $(x_i, y_i, 0)$ and $(x_i, y_i, 1)$		N, state of cells at $(x_i, y_i, -1)$ and $(x_i, y_i, 2)$	
N	Next state	N	Next state	N	Next state
0	dead	0	dead	0	dead
1	dead	2	dead	1	dead
2	dead	4	dead	2	dead
3	alive	6	alive	3	dead
4	dead	8	dead	4	dead
5	dead	10	dead	5	dead
6	dead	12	dead	6	alive
7	dead	14	dead	7	dead
8	dead	16	dead	8	dead

Table 2: Here we are concerned about next generation status for cells that are not alive, but are in the immediate vicinity of live cells. When the Conway object contains vacant cells with six live neighbors, then the next generation of the Life 5766 expansion will have new live cells in planes adjacent to the expansion.

of them satisfy the criteria of the above theorem (see figure 4). Conway's glider has an analog under (5766), as do some of the more complicated oscillators. Unfortunately, many of the more interesting and important Conway objects do not have analogs; for example, there is no analog for the "glider gun" and other so-called breeding oscillators.

Preliminary testing has revealed that collisions between gliders and other objects, though occasionally analogous, usually yield non-analogous results. Sometimes the first few generations after impact of analogous objects behave nicely, but sooner or later the conditions of theorem 7 are usually violated; when this happens, the object, theretofore confined to two planes, almost always forms a roundish three-dimensional mass that usually dies rather quickly, but occasionally stabilizes.

Note that analogous behavior is very narrow in scope—it takes place entirely in two adjacent parallel planes. Obviously, we may alter any three-dimensional glider–object collision by shifting one of the participants in the Z direction. Thus, if the analogs lie in the $Z = 0$, $Z = 1$ planes, we can shift one of the objects by one, two, or three Z planes and achieve entirely different collision results. Furthermore, we need not confine our objects to nearby Z planes—a glider can, after all, attack from a perpendicular plane (see figure 6). Hence, analogous behavior is at most a small subset of Life 5766, which is replete with its own objects and collisions.

It is, of course, rather convenient to have an immediate supply of known stable and oscillating Conway analogs already available for Life 5766. Furthermore, we will soon see that it just might be possible to construct a three-dimensional glider gun by placing appropriate objects on either side—

most likely in the $Z = -2$ and $Z = 3$ planes.

2.2.1 Time-space barriers

If we build a stable planar form where each living cell has seven neighbors, then no births can ever occur in the two parallel adjacent planes; these planes are "dead". A portion of such a form, called a *time-space barrier*, is shown in figure 7. If the flat part lies in the $Z = 0$ plane (extending an unspecified amount in the positive X and Y directions), then no glider or other form approaching from a higher Z plane can ever penetrate into the $Z = 1$ plane. Of course, the same is true on the other side of the barrier, and we have not considered the boundary, which here has been stabilized with eight-element cubes. Naturally, we might choose to have our barrier extend to infinity in the X and Y directions.

2.2.2 "Nearly 3-D" Life and Conway's game as a subset

Construct two arbitrarily large parallel time-space barriers and place them initially quite some distance apart. Life forms under $R = (5766)$ would behave in their usual unrestricted fashion insofar as our distant barriers would allow. But now we will move the barriers closer together. As we do, evolving Life forms would be "squeezed"; growth in the Z direction becomes more and more inhibited. For example, when the barriers are separated by six planes, all life is confined to the four planes in the middle—what we have here is a "nearly 3-D" Life where each spacing of the barriers exhibits a version of the game whose behavior is distinct from any other configuration. Now consider what happens when the barriers are four planes apart (figure 7). All life must then be confined to two planes. Recall from theorem 7 that the (5766) analog to Conway's Life breaks down only because of growth in the Z direction. But now we have prevented such growth; hence, *an analog to the entire Conway Life universe is contained between the barriers.* For that matter, we could construct an infinite number of parallel Conway Life universes.

Of course, nothing can slip out (in the X or Y directions) from between finite barriers—at least as they are constructed. For example, the edges would interact with any escaping glider, thus ruling out a simple glider gun. Possibly, some oscillators could be appropriately placed to allow a glider to leave the vicinity unimpaired; this is undoubtedly the easiest approach to gun construction.

2.3 The rule $R = (4555)$

We can build charts similar to tables 1 and 2 for $R = (4555)$, yielding the following results.

Theorem 9. *A stable Conway object has an analog under $R = (4555)$ if and only if each living cell in the Conway object has exactly two neighbors.*

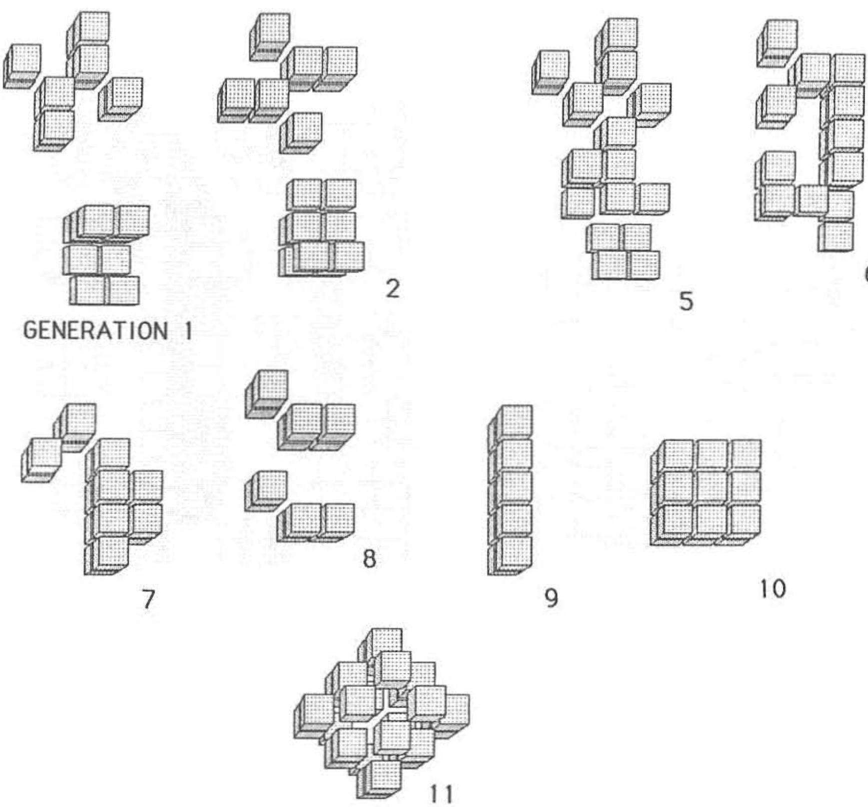

Figure 6: A collision between objects that are Conway analogs in Life 5766—a glider and a "clock". Here, however, the glider is attacking from a perpendicular plane. As far as the entire configuration is concerned, only generations 7 through 10 are analogs of Conway Life. At generation 10, theorem 1 part b, is violated; hence, growth in the Z direction is initiated. For this particular collision, the living mass seemed to remain confined to two planes for a short while before suddenly "releasing" a stable 24-element object. The usual result of a collision is a rather quick annihilation of both objects.

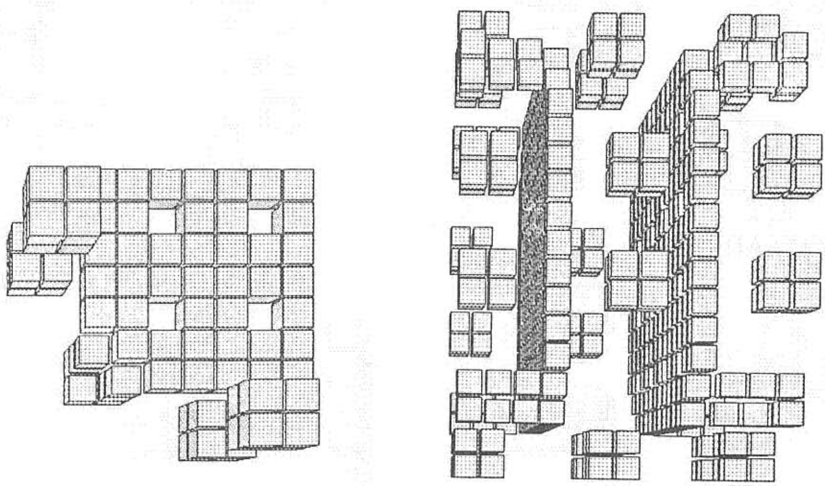

Figure 7: The object at the left is a portion of a "time-space barrier". No life can get within one cell of the flat portion of this form, which can be made to extend indefinitely. The finite parallel barriers at the right have four planes between. Hence, a "mini Conway universe" analog can exist in the two middle planes as long as no shape wanders too close to the barrier boundaries.

Corollary 10. *A Conway object that changes from one generation to the next has no analog under the three-dimensional Life rule $R = (4555)$.*

The conclusion is that although $R = (4555)$ has an occasional analogous form or two, its "universe" is completely different from that of Conway—and, for that matter, different from the universe of $R = (5766)$. It is interesting that although the two three-dimensional Life games behave in totally different ways, they both seem to stabilize rather quickly, with (4555) requiring somewhat more time. For example, if we start with a random configuration in, say, a $70 \times 70 \times 70$ grid, both rules will stabilize after about 30 to 70 generations, depending upon the starting density. Collisions between gliders and other small objects usually die after about 5 to 20 generations—unless they happen to yield debris.

The early discovery of a totally distinct glider (figure 8) was the catalyst that led to the extensive investigation of this rule. The (4555) glider contains ten elements and, like Conway's glider (and its (5766) analog), has a period of four after which it has moved a distance of $\sqrt{2}$ in one of twelve directions—perpendicular to one coordinate axis and at an angle of 45 degrees with the other two.

2.3.1 Additional shapes of Life 4555

The relative low density of stable life (resulting when we start with pseudo-random primordial soup) is more than compensated for by the rich variety and symmetry of small Life forms. Several of these "naturally" occurring forms are shown in figure 9.

We may create primordial soup in several ways. Perhaps the easiest method is to initialize each cell in our universe according to the output of a random number generator. For example, on our exhaustive journey through the universe, as we pass each cell generate a random number, r. Then, for some fixed constant, k, if $r < k$, let that cell be alive, otherwise not.

We can get some idea of the relative paucity of Life forms by examining table 3. The entries were found by applying the "soup stirring" rule $(4512)_g$ repeatedly g times to a $70 \times 70 \times 70$ space that had been filled with about 40 random living cells. The rule (4555) was then employed. Ten samples were made with g set to various values between 14 and 27. All forms were allowed to stabilize; this usually occurred after about 60 or 70 generations. The stable residue was then tallied. The lone observed glider was not counted. Perhaps gliders are more common than this table would indicate, as they may have gone off the screen (or collided with something) before being observed.

2.3.2 Glider collisions in Life 4555

The huge number of reflections and rotations of the small stable forms leads to a myriad of distinct possible collisions between gliders and other

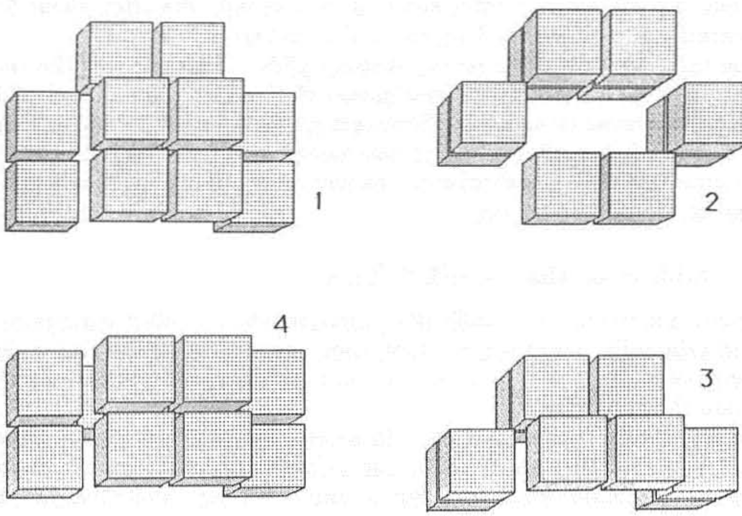

Figure 8: The (4555) glider, showing the four states. When state one is encountered again, the glider will have moved one unit up and one forward (i.e., in the positive YZ direction).

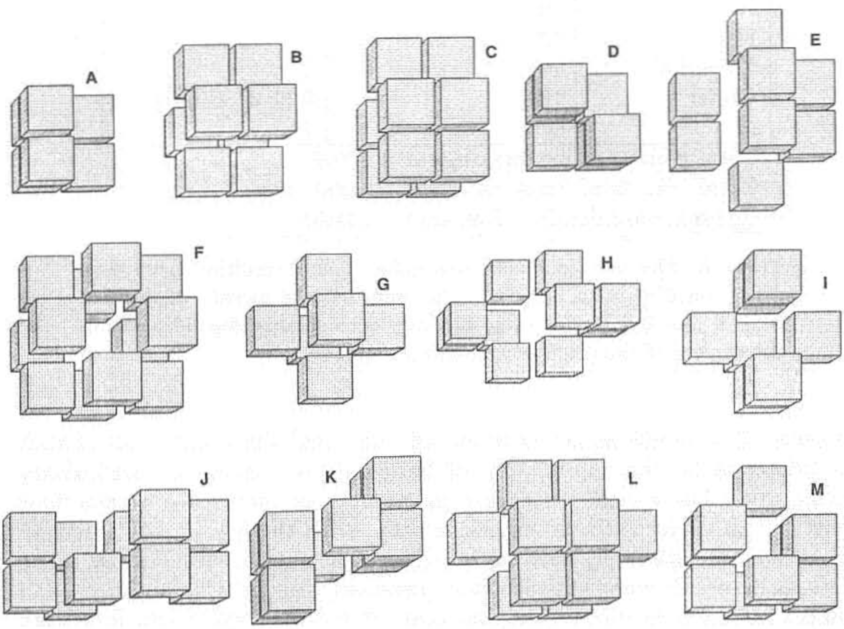

Figure 9: A few of the small symmetric forms which occur "naturally" in Life 4555.

Object (see fig. 9)	Number of elements in the object	Occurrences of object at stability
A	6	124
B	8	37
C (period = 4)	8 or 10 (ave. = 9)	36
D	6	28
E (period = 4)	8 or 10 (ave. = 9)	14
F	12	3
G	6	3
H	10	2
I (period = 2)	7	2
J	10	1
K	10	1
L (period = 4)	10	1
Glider	10	0 (1 observed)
M	9	1

Total volume of residue objects = 1,761
Total volume of "universe" = 3,430,000
Approximate density of residue = .00051

Table 3: The most common Life 4555 objects resulting from condensation of primordial soup. The approximate density of the live residue was .0005. This value can vary considerably depending upon the density of the original primordial soup.

objects. One would expect (and indeed one finds) the usual result of such collisions to be the annihilation of both objects. However, preliminary exploration has revealed a surprising number of interesting interactions (see (1) Appendix A for an extensive list). With the low ($\sim 10^{-3}$) density of stable life ultimately settling out from "soup", the fact that so many Life forms result when a glider (ten elements) collides with another small object (about ten elements) implies that some rather mysterious forces are at work. A typical interesting collision result is shown in figure 10.

2.3.3 Manufactured stable forms

Although most of the stable Life 4555 shapes found "in nature" (i.e., as the result of some evolving population, randomly created or otherwise) rarely contain more than about a dozen elements, it is possible to construct exotic stable forms (see figure 11). These forms would be highly unlikely to appear as the result of a primordial soup experiment. One should note that such forms are harder to construct for Life 4555 than for Life 5766; this is due to the more limited safe environment range.

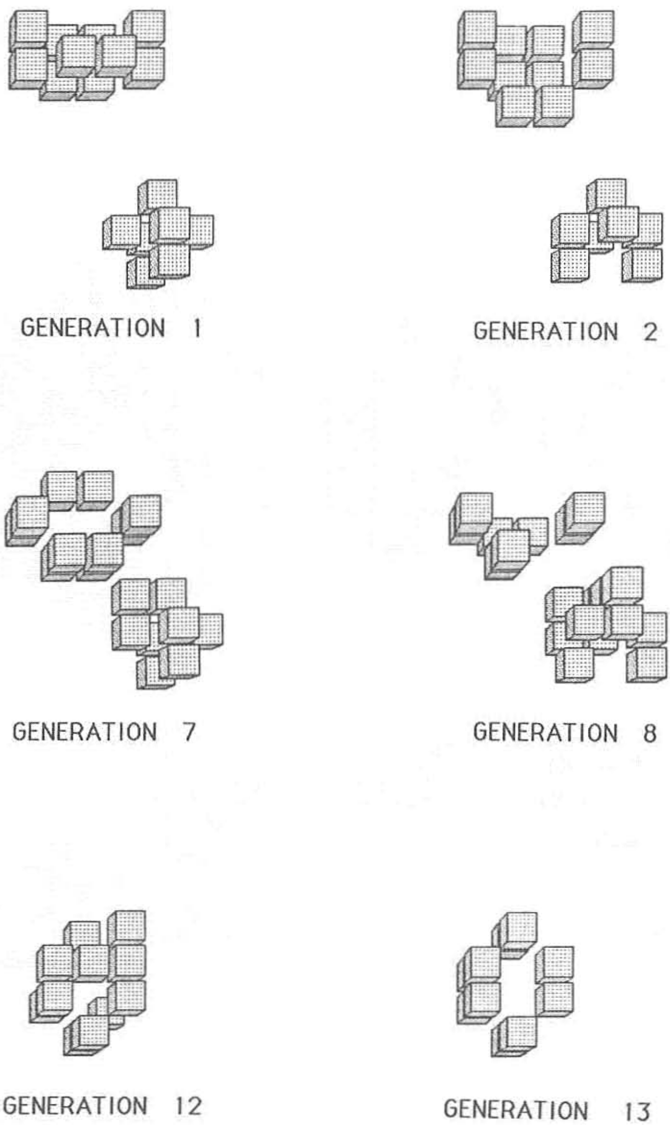

Figure 10: One of the more interesting Life 4555 glider collisions. Here, a glider collides with an object called a "blinker". The original glider and the blinker are destroyed, but a new glider appears. If the original glider was traveling in the $(-Y - Z)$ direction, then the new one will be heading toward $(-X + Z)$.

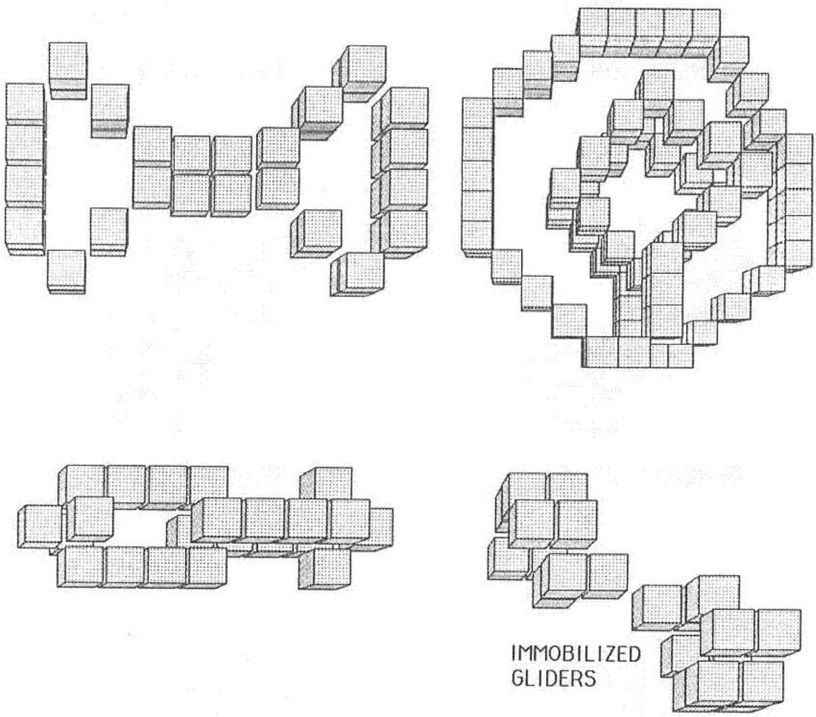

Figure 11: Manufactured Life 4555 stable forms such as these would never be found by conducting primordial soup experiments; they must be carefully constructed.

3. Other possibilities

We may broaden our allowable configurations somewhat by treating corner, edge, and side neighbors as distinct types. Hence, we have six "face" neighbors, twelve "edge" neighbors and eight "corner" neighbors (see figure 12). For the Life game (whether in two or three dimensions), all neighbors have equal weight, whether touching on a face or a corner. We introduce the following notation. Let R denote any rule of the form $E_l E_u F_l F_u$. Then, Life.100 R denotes a neighborhood consisting only of the eight corner neighbors; Life.010 R denotes a neighborhood consisting only of the twelve edge neighbors, and Life.001 R similarly deals only with the six face neighbors. Thus, Life 4555 can also be written Life.111 4555, etc.

Two additional gliders have been found—one in Life.011 4544 and another in Life.110 4544. These are the only rules that seem to support gliders. (Here, we do not consider rules such as Life.x 1111, which provably allows infinite growth.) Unfortunately, primordial soup experiments with Life.110 4544 and Life 011.4544 usually lead to unbounded growth; hence, these rules have not been investigated further.

It is very important to note that if we expand our rules to consider the more general neighborhood configuration (there are 2^{26} not counting reflections and rotations), then other gliders probably exist and definition 1 can likely be satisfied—but the beauty of Life is its simplicity. The next section discusses the most elegant rule of all.

4. Another Game of Life

Much energy has been expended in an effort to discover a Life game in the two-dimensional hexagonal grid, where each cell has six neighbors. Unfortunately, no worthy rule exists, unless we consider exotic configurations such as Golay surrounds [5].

But let us expand the two-dimensional hexagonal grid into three dimensions. We then obtain the hexahedral tessellation, a universe where neighbors can be represented by the corners of the 14-sided hexadecahedron (figure 13). Here, there are 12 neighbors which line up in four intersecting hexagons. These hexagons form four non-orthogonal planes which are parallel to the sides of a regular tetrahedron. This configuration can also be represented by "densely packed spheres" and conforms to certain natural crystal structures.

Note that there are two distinct universes, U and U_r (see figure 13). For the remainder of this discussion, we shall deal only with U; U_r is obtained by reflecting U in a vertical plane. Also, we shall use spheres to represent cells, although we could also use hexadecahedrons, or, more simply, points with neighbors connected by lines of equal length. Again, although there are 2^{12} possible neighborhood configurations, we shall only consider the quantity of neighbors and not their orientation. To narrow down the possible Life rules, note the following theorems.

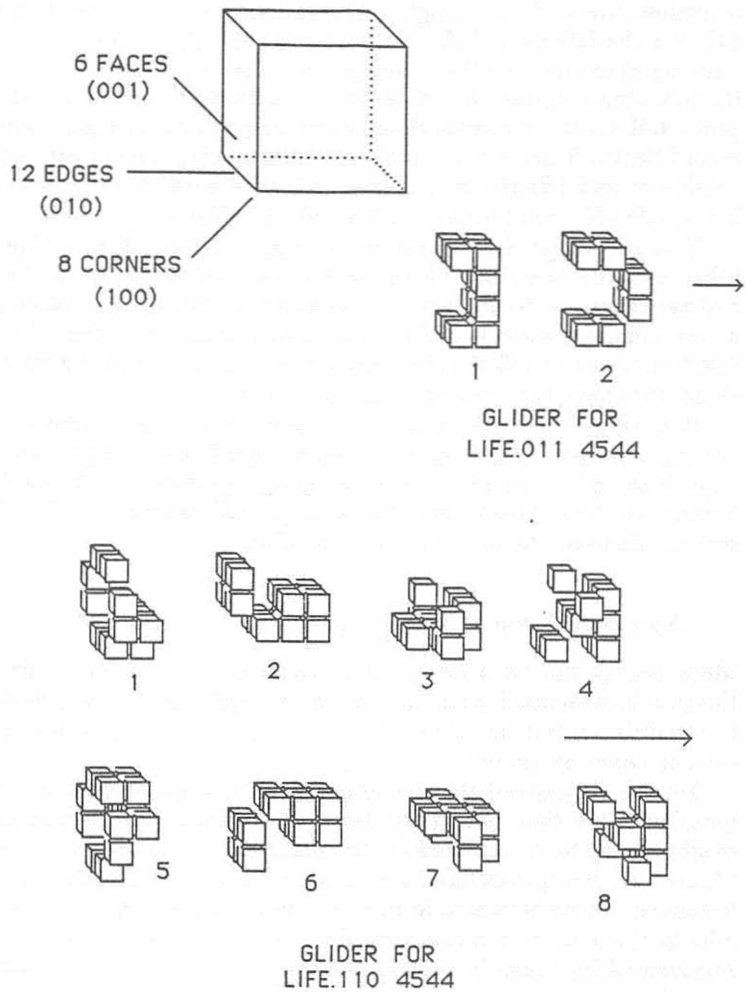

Figure 12: Two gliders have been discovered for life rules that would be "worthy of the name," except primordial growth under these rules is unlimited.

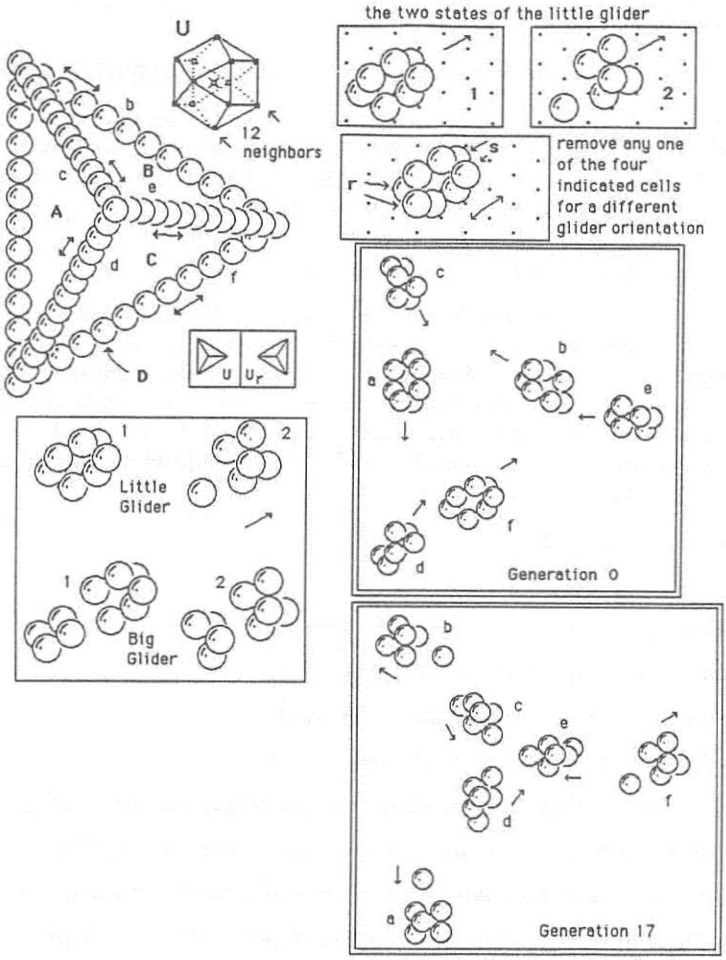

Figure 13: (Clockwise from upper left) The skeleton of a tetrahedron shows the four planes (A–D) that are parallel to the four hexagonal neighborhoods in U. U_r is obtained by reflecting U in a vertical plane. The two universes are distinct. The little glider has two states and travels parallel to an edge (a–f). By removing any one of the cells (r,s), we obtain a different orientation; hence, there are 24 different orientations. The six gliders at generation zero will travel parallel to edges (a–f); at generation 17, they have advanced as indicated. As it advances, the 10-element big glider resembles a frog. The 7-element little glider is the "tadpole".

Theorem 11. *Any rule $E_lE_uF_lF_u$ where $F_l \geq 4$ cannot support a glider.*

The proof is analogous to that of theorem 3, and the theorem below corresponds to theorem 2.

Theorem 12. *Any rule $E_lE_uF_lF_u$ where $F_l \leq 2$ allows unlimited growth.*

Thus, if any Game of Life exists, it must be of the form E_lE_u33. The only rule that seems to exhibit gliders is (rather nicely) $R = (3333)$ (see figure 13). This rule also has bounded growth; hence, our definition of a Life game "worthy of the name" has been satisfied.

4.1 Symmetry of objects in Life 3333

One of the most interesting features of Life 3333 is that every stable or oscillating object so far discovered exhibits symmetry in some form. For convenience, consider the tetrahedron in figure 13, itself constructed of cells in U. (By the way, this tetrahedron is not a stable structure in Life 3333.) The four axis planes correspond to sides A, B, C, and D; the edges mentioned below refer to a, b, c, d, e, and f. So far, objects discovered have exhibited symmetry about:

1. a point (six-way)
2. a point (four-way)
3. a line perpendicular to a side (six-way)
4. a line perpendicular to a side (three-way)
5. a line perpendicular to a side (four-way)
6. a line perpendicular to a side and the side
7. a line perpendicular to two opposing edges (e.g. edges a and e)
8. a line described above and a plane perpendicular to this line
9. a plane parallel to a side and perpendicular to the opposing side
10. a plane described above and a plane perpendicular to this plane

When figuring the symmetry of a periodic object, we must consider whether object phases are reflections of each other, rotations of each other, and so on. We need to find the symmetry type in order to determine how many different orientations of the object exist. For example, if an object exhibits six-way symmetry about a point, then there is only one distinct orientation. On the other hand, if a totally asymmetric object were found (so far it has not been), it would have 48 distinct orientations. Objects with distinct orientations numbering all the factors of 24 have been found (see figure 14).

Life 3333 is currently under intense investigation and may ultimately be revealed as the most interesting Life game of all.

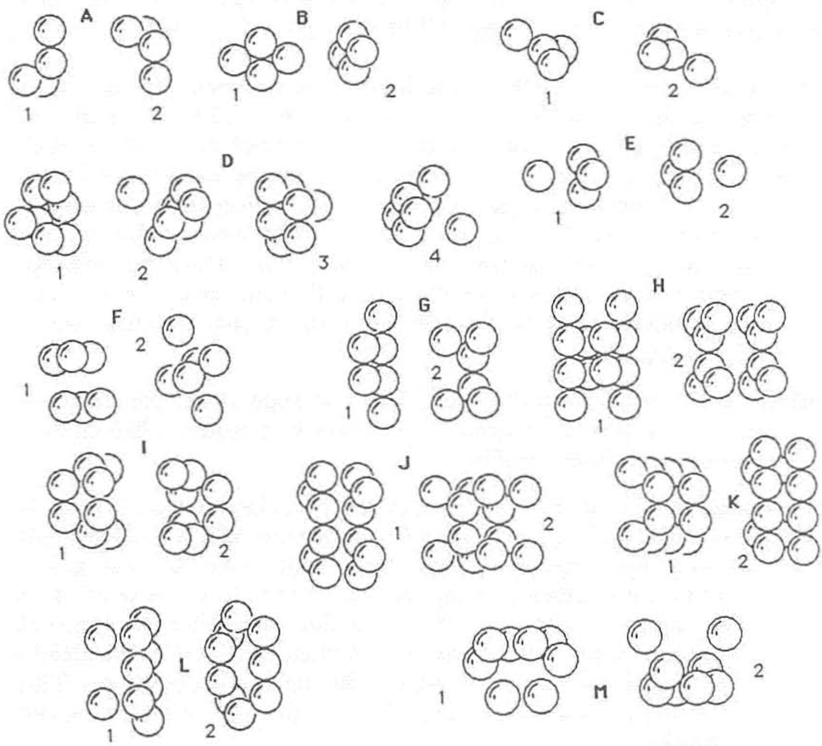

Figure 14: Here are just a few of the many objects in Life 3333. All that have been discovered so far exhibit symmetry of one form or another. Period two oscillators abound, but oscillators with other periods have been found. Motionless stable objects seem to be quite rare.

5. Programming the Games of Life 4555 and Life 5766

For the small computer, two types of algorithms may be employed. More exotic approaches probably can be found, but the methods below are reasonably straightforward. The discussion here uses $R = (4555)$; $R = (5766)$ could just as easily be employed. The non-orthogonal Life 3333 presents a special programming problem and will be deferred to later work.

Method A. Tally living cells in the leading nine-element "plane" as it sweeps across space. Thus, at a given cell, we need to make only 11 calculations instead of 26. Essentially, as we check neighbors for each cell, we add the living cells found in leading nine elements, subtract the trailing nine, and adjust the center. This concept can be carried one dimension further, tallying a leading three-element line in each nine-element plane; however, we only save about three calculations. Method A is useful when developing starting shapes or creating primordial soup. It also can be used when the density of living cells is relatively high.

Method B. This method utilizes the fact that populations are relatively sparse. Keep within each cell: (a) whether it is dead or alive and (b) the number of living neighbors.

Option 1. As we are examining each cell in the universe, if a cell is nonvital and has fewer than five neighbors, take no action. This can be done with one test (i.e., represent "alive" by adding, say, 32 to the number of neighbors. Our loop then checks whether a cell is less than five). We then check for other less frequent situations. If a cell is to die, subtract one from each of its neighbors, and if a new cell is born, add one to each neighbor. This method will speed up or slow down as the population grows and shrinks.

Option 2. Keep the living cells sorted in some fashion (or in a hash table for quick access). Then, we need only examine *changed* cells and alter next-generation states where appropriate while we are tallying neighbors. Hence, if we find that a new cell is born, we immediately place it in the list of changes for the next generation. Similarly, if a cell dies, we place it on the list of changes indicating that it is to be removed. After the entire universe has been examined, we merge the "changes" (adds and deletes) with the prior generation's living cell list. This algorithm should use hash tables for the changes and should keep track of the neighborhoods and whether a cell is alive, in a three-dimensional array (sparse or otherwise). Note that our three-dimensional universe array can be as large as memory allows, with little degradation to the execution time. Furthermore, if we use sparse hash arrays, an infinite universe is possible. This algorithm is rather tricky to implement, but can be exceedingly fast. It has been implemented in C on the Ridge 32 and the Macintosh computers and evaluates most patterns at about 4 to 15 generations per second, depending upon the pattern size.

5.1 Execution speed of the algorithms

(For all the cases about to be discussed, we have ignored the time required to display the cubes.) Let N be the size of the universe cube edge and let p be the number of living objects. We should note that a "worst case" but trivial algorithm simply looks at every cell and tallies the neighbors. This algorithm runs in time $26kN^3$, where k is some constant that depends upon the time required on a particular computer to fetch an element from a three-dimensional array. Method A reduces this time to $11kN^3$. Method B option 1 increases the speed to about $kN^3 + 26k_1p$, where $k_1 \approx 2k$. Since $p \ll N^3$, we obtain a significant increase in speed. Method B option 2 further enhances the speed by eliminating the N^3 term, achieving approximately $26k_1p + k_2p$. Here, k_2 is somewhat larger than k and depends upon the hashing methods employed. A further characteristic of this method is the fact that a particular cube may come and go in the "changes" list several times during evaluation of a given generation before its final status (alive or dead) is determined. Experience has indicated that for Life 4555 there are usually three times as many references to the "changes" hash table during generation evaluation as there are actual changes that finally get merged and plotted at the end of that particular generation evaluation. Some sort of exotic parallel architecture could avoid this.

6. Summary and conclusion

The two rules for three-dimensional Life "worthy of the name" in the orthogonal universe have been presented and investigated. The rule (5766) relates closely to Conway's two-dimensional Life—to the extent that *any* three-dimensional rule could. Many of Conway's shapes have analogs under (5766); furthermore, Conway's entire universe, or an infinite number of parallel universes, can be contained with time-space barriers.

The rule (4555) yields a distinct rich universe of small symmetric stable and oscillating forms. This rule has little in common with either Conway's Life or the rule (5766). Other rules such as (5655) and (4666) are interesting in that they support small symmetric oscillators, but they do not fit definition 1.

6.1 Possible limitations

There are two important differences between the behavior-at-large of Life forms in two-dimensional Life and three-dimensional Life. Stable residue eventually resulting from appropriately primed "random primordial soup" occupies about 5 percent of Conway's universe, but only about .05 to .1

percent in a three-dimensional Life universe. This paucity of population seems to be compensated for by the large number of small stable forms, their reflections and rotations; hence, interesting interactions are abundant. Life forms, though sparse, are intense.

The second difference, unfortunately, may ultimately be a factor that prevents either three-dimensional Life rule from yielding exotic constructs similar to Conway's huge "puffer train" or "spaceship factory". Many configurations in Conway's Life allow a large number of generations to elapse before settling down. (For example, the seven-element "acorn" evolves to a maximum of 1057 living cells and finally stabilizes at generation 5206.) Hence, the possibility of complicated intermediate interactions. Most three-dimensional Life interactions, on the other hand, converge rapidly—more so under (5766) than (4555). (Moreover, the line between converging and diverging rules is a fine one; e.g., $R = (5755)$ exhibits unlimited growth.) This is not meant to imply that, for example, a glider gun does not exist in three dimensions. In fact, under (5766), it is likely that one might be constructed by stabilizing the emerging glider from a two-dimensional gun analog that has been confined with a time-space barrier. Such a construction would probably consist of carefully orchestrated oscillators situated at the barrier edge near the emerging glider.

On the other hand, a "native" glider gun under either (5766) or (4555) will be difficult to discover—the task appears more difficult for (5766) Life since a non-analogous glider does not appear to exist. Of course, a three-dimensional gun of *any* type would open the universe to the construction of sparse but purposeful Life forms.

The recent discovery of Life 3333 in the hexahedral tessellation is currently undergoing intense investigation and, due to the highly symmetric nature of its objects (and the relevance to inorganic crystal structure), may eventually turn out to be the most interesting three-dimensional Game of Life "worthy of the name."

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