Franklin Squares: A Chapter in the Scientific Studies of Magical Squares

Peter D. Loly*

Department of Physics and Astronomy,
The University of Manitoba,
Winnipeg, MB Canada R3T 2N2

Several aspects of magic(al) squares studies fall within the computational universe. Experimental computation has revealed patterns, some of which have lead to analytic insights, theorems, or combinatorial results. Other numerical experiments have provided statistical results for some very difficult problems.

While classical nth order magic squares with the entries $1..n^2$ must have the magic sum for each row, column, and the main diagonals, there are some interesting relatives for which these restrictions are increased or relaxed. These include: serial squares of all orders with sequential filling of rows which are always pandiagonal (i.e., having all parallel diagonals to the main ones on tiling with the same magic sum, also called broken diagonals); pandiagonal logic squares of orders 2^n derived from Karnaugh maps; Franklin squares of orders $8n$ which are not required to have any diagonal properties, but have equal half row and column sums and 2-by-2 quartets; as well as sets of parallel magical bent diagonals.

Our early explorations of magic squares, considered as square matrices, used *Mathematica*® to study their eigenproperties. We have also studied the moment of inertia and multipole moments of magic squares and cubes (treating the numerical entries as masses or charges), finding some elegant theorems. We have also shown how to easily compound smaller squares into very high order squares. There are patents proposing the use of magical squares for cryptography. Other possible applications include dither matrices for image processing and providing tests for developing constraint satisfaction problem (CSP) solvers for difficult problems.

1. Introduction

The present contribution comes on the heels of the landmark count of Franklin squares on a chessboard by Schindel, Rempel, and Loly [1]. Paul Pasles [2] has provided a beautiful historical context for the resurgence of interest in Franklin squares, and Maya Ahmed [3, 4] posed the question that lead to our effort to count the number of Franklin

*Electronic mail address: loly@cc.umanitoba.ca.

squares which used the full set of elements 1..64. This work has already been reviewed by Ivars Peterson [5].

Small magic squares are often encountered in early grades as an arithmetic game on square arrays (patterns or motifs). Classical magic squares (hereafter just referred to as magic squares) of the whole numbers $1 \dots n^2$ have the same line sum (magic constant) for each row, column, and main diagonal:

$$C_n = \frac{n(n^2+1)}{2}. \tag{1}$$

This line sum invariance depends only on the order n of the magic square. The ancient 3×3 Chinese Lo-shu square of the first nine consecutive integers is the smallest magic square and, apart from rotations and reflections, there are no others this size or smaller:

4	9	2
3	5	7
8	1	6

(2)

where the magic sum is 15. For $n = 3$, $C_3 = 15$, as expected from equation (1). The best statement that can be made about its age is 2500 ± 1500 years! While the middle figure may be most appropriate, the hardest evidence gives just the lesser [6], while legends [7] claim the older.

We briefly review the recreational aspect of magic squares before drawing attention to the simpler semi-magic squares, and to pandiagonal nonmagic squares. These we lump together under the rubric of magical squares. Then we examine the scientific aspects of all these squares through applications in classical physics and matrix analysis. In fact it was partly through the coupled oscillator problem that the mathematics of matrices was developed. Through an elementary example in matrix-vector multiplication, which can be done at the high school level, we demonstrate a simple eigenvalue-eigenvector problem. Magic squares can then play a valuable role in modern courses in linear algebra.

2. Recreational mathematics

At the recreational level magic squares are fun for all ages, as I found when introducing them to visitors during the summer of 2000 whilst volunteering at the "Arithmetricks" traveling exhibit at the Museum of Man and Nature in Winnipeg, Manitoba. Various types of magic squares have become a recreational pastime of amateurs, often very gifted individuals, for example, Albrecht Dürer and Ben Franklin. In 2006 I gave talks over a wide range of grade levels on aspects of magical squares.

While there are several journals which publish results in this area of recreational mathematics, the rise of the World Wide Web now affords many of the actors a place to publish extensive work without recourse to the oft tedious rigors of peer review, the selectivity of editors, and a considerable time delay. There are some real gems from these efforts, but there are obvious drawbacks for the longer term flowing from absence of refereeing and lack of permanent archiving.

However magic squares present difficult challenges for mathematicians and over the past few hundred years many famous mathematicians have contributed to our knowledge of them, including Euler.

Consider the number of possible arrangements of 1..9 in a 3×3 square after removing an 8-fold redundancy factor from rotations and reflections, $9!/8 = 45{,}360$. Constraints on row, column, or diagonal line sums sharply reduce the number of squares, and there is plenty of room for other types of alternative constraints to produce interesting squares. For ninth order Latin squares where 1..9 appear once in each row and column, the current fad of Sudoku has plenty of scope when those entries are also required to appear once in the nine subsquares.

3. Art and design: Line paths in magic squares

This probably goes back at least a century to a time when there were many fewer magic squares than are available today. Early in the twentieth century the famed architect Claude Fayette Bragdon [8] used line paths from some order 3, 4, and 5 magic squares as the basis of ornamentation for the interior and exterior of buildings, especially in Rochester, New York. Bragdon also shows line paths for Franklin's $n = 16$ square (see also Clifford Pickover's recent book [9], where he includes one from Ben Franklin). If one considers the 880 fourth order magic squares enumerated in 1693 by Frénicle de Bessy [10, 11], which have been classified by Dudeney into 13 types, then the symmetry of the line path may distinguish squares with more than the minimal constraints.

4. Where is the science?

Why did a theoretical physicist get involved with magic squares? This was not a linear process. Having been blissfully unaware of them for my first five decades, an encounter with the Myers-Briggs Type Indicator© (MBTI) [12] scheme of personalities, presented as a 4×4 design, resonated with my background in mathematical structures, partly from research in crystal physics. Coordinate rotation matrices in classical and relativistic mechanics, combined with periodic boundary conditions for finite crystals soon lead to links with magic squares, while the psychological nature of the MBTI eventually made connections with early work of Carl Jung, which finally connected with Jung's [13] interest in Chinese patterns, especially the Lo-shu magic square, the dichotomous yin-yang schemes of Feng Shui (a.k.a. *The Golden Flower*), and the I Ching.

The science begins whenever we go beyond a recipe approach for constructing a single square, for example, counting or estimating the populations of various classes of squares, proving that none are possible in a given case, interpreting them as arrays of point masses, or as electric charges. There are more examples, but another early aspect of our own work focusses on the remarkable results associated with treating magic squares as matrices in the context of linear algebra, that is, solving sets of simultaneous equations, as well as some topics in coupled oscillator physics.

5. Some special varieties of squares

For a number of purposes it is important to recognize that there are several specially important variations on the theme of magic squares.

Semi-magic squares, do not necessarily have the diagonals summing to the row-column line sum of equation (1), some may be obtained simply by moving an edge row and/or column of equation (2) to the opposite side, for example,

9	2	4
5	7	3
1	6	8

(3)

The removal of the diagonal constraints means more squares due to the smaller number of constraints, in this case there are eight more.

Pandiagonal nonmagic squares, have the same magic line sum for all the split lines parallel to the main diagonals. We can illustrate pandiagonals by taking a nonmagic serial square (having the consecutive integers fill row-by-row) and tiling a copy to its right (or left, or top, or bottom):

1	2	3	1	2	3
4	5	6	4	5	6
7	8	9	7	8	9

(4)

The pandiagonals are $(1,5,9)$, $(2,6,7)$, $(3,4,8)$, $(3,5,7)$, $(1,6,8)$, and $(2,4,9)$, together with the main diagonals. Observe that for a given order the number of row and column constraints is the same as the number of pandiagonal constraints. Serial squares exist for all orders, unlike magic squares which do not exist for $n = 2$.

A pandiagonal magic square is the combination of this pandiagonal property with the requirements of a magic square. These first occur in order four, and of these 48 are found amongst the fourth order squares, with none possible for singly even orders $(6, 10, \ldots)$:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(5)

Another interesting type, namely *associative* (or *regular*) *squares* have the antipodal property:

$$a_{ij} + a_{n-i+1,n-j+1} = n^2 + 1; \qquad i, j = 1..n. \tag{6}$$

Finally, our recent Franklin result [1], and the *most-perfect pandiagonal magic* (or *complete*) *squares* of McClintock [14], for which Ollerenshaw and Brée's [6] combinatorial count ranks as a major achievement, draw attention to another type which have the same sum for all 2×2 subsquares (or quartets). *Franklin squares* also have constant half row and column sums, and constant parallel bent diagonals [1].

Ollerenshaw and Brée [15] have a patent for using most-perfect magic squares for cryptography, and Besslich [16, 17] has proposed using pandiagonal magic squares as dither matrices for image processing.

6. Counting magic squares: The role of backtracking computations

There are 880 distinct 4×4 magic squares of the first 16 integers, and 275,305,224 distinct 5×5 squares of the first 25 integers, the latter were first counted by computer in 1973 [18]. Already by order six they have become far too large to count at present, and as a result only statistical estimates are possible. Pinn and Wieczerkowski [19] performed a Monte Carlo simulated annealing computation for an estimate of $(0.17745 \pm 0.00016) \times 10^{20}$ for the 6×6, and a less accurate estimate of the 7×7.

Walter Trump [20] developed a more efficient hybrid backtracking Monte Carlo method and improved the accuracy of these estimates. He has also made good estimates of the number of various types of magic squares up to 10×10, a remarkable achievement. During summer 2003 two of my undergraduate students at Manitoba took Trump's 6×6 GB32 code, which uses 13 random cells, and converted it to C++. Matt Rempel found that the C++ code runs about 10% faster than GB32 on the same PC in producing a sample of some 700,000 squares, and has begun to remove random cells, finding longer run times, but more accurate results, and a larger sample of magic squares. Dan Schindel took the ideas in the 6×6 code and constructed pure backtracking codes without random cells to count the known numbers of magic squares for the 4×4 and 5×5 cases. While we have previously been able to analyze the complete 4×4 set, we can now begin to analyze the 5×5. This gives us the ability to study a variety of interesting questions, for example, their eigenproperties. Dan Schindel has also amended the 4×4 code to count the number of pandiagonal nonmagic 4×4 squares, finding some three million.

There are some Java backtracking codes which can be run directly from web pages. Meyer [21] has one of the best and it will find a stream of different squares for $n = 4$, 5, and 6.

A simplified demonstration of counting the 880 fourth order magic squares using *Mathematica* is given in [22]. Our *Mathematica* code can be refined and we hope to harmonize it with Eric Weisstein's *Mathematica* tools for magic squares [23].

7. Integer points in polyhedral cones

Ahmed [3] asked "How Many Squares Are There, Mr. Franklin?. . . " in a paper which exploited Hilbert bases of polyhedral cones (PHC) to construct several new natural Franklin squares. Ahmed [3, 4] was also able to use PHC techniques to count the total number of eighth order Franklin squares as a function of a variable magic sum (s), obtaining a count of 228,881,701,845,346 for $s = C_8 = 260$. However this includes many squares with degenerate elements, so that this count is an upper bound to the population of natural squares. PHC techniques do not at present permit the elements to be distinct, so they do not give the smaller counts expected for natural squares, where the elements $1..n^2$ are distinct. These PHC techniques, and their upper bound statements, have received considerable attention recently [24], in publications (e.g., Ahmed [3, 4]; Ahmed, De Loera, and Hemmecke [25]; and Beck *et al.* [26]), and have recently been the subject of advanced courses [27, 28].

8. Franklin squares

Our recent exact count of eighth order Franklin squares [1] used a modification of Walter Trump's backtracking strategy to perform the Franklin count (the source and a datafile of the 1,105,920 distinct squares are also available [1]). During the NKS 2006 conference I received a message from Miguel Amela [29] enlarging on this work by using ideas of Chebrakov [30].

9. Comparison of various counts

We constructed Table 1 after the fashion of Trump [20] to collect some extant results for relevant counts and to summarize some of the major results.

order of square, n	4	5	8
natural magic sum, C_n	34	65	260
natural magic	880	275,305,224 (a)	$5.2210(70) \cdot 10^{54}$ (b)
associative natural	48	48,544 (b)	$2.5228(14) \cdot 10^{27}$ (b)
natural panmagic	48	3,600 (b)	(c)
complete	48	-	368,640 (d)
natural Franklin	0 (e)	-	1,105,920 (f)
PHC upper bounds			
pan Franklin (g,h)	-	-	10,308,923,109,408
Franklin (g,h)	-	-	228,881,701,845,346
magic (i,j)	163,890,864		
panmagic (h)	35,208	53,852,072,626	

Table 1. Comparison of various counts: (a) Schroeppel [18]; (b) Trump [20] with statistical errors for $n = 8$; (c) lies between preceding numbers in this column from (b); (d) Ollerenshaw and Brée [6]; (e) Pasles [2], see footnote 22 on page 506; (f) Schindel *et al.* [1]; (g) Ahmed [3]; (h) Ahmed [4]; (i) Ahmed, De Loera, and Hemmecke [25]; and (j) Beck [26].

10. Magic squares and constraint satisfaction problem solvers

CSPs fall into several classes, for example, genetic algorithms and evolutionary computing. Magic squares are frequently used as test targets in benchmarking improved CSP strategies. Sometimes it is the time to find the first square [31], or the time to find a complete set such as the 48 fourth order pandiagonal (also most-perfect) squares [32]. Hnich and Walsh [33] studied the time to find the first solution for $n = 3..6$ using several permutation models. The highly constrained eighth order most-perfect squares have also been used by Roney-Dougal *et al.* [34] with symmetry reduction to find all the principal reversible squares of Ollerenshaw and Brée [6] for $n = 4$, 8, and 12. Also, Xie and Tang [35] report timing for 10 examples for orders 10 (10) 100.

11. Compound squares

An undergraduate project with Wayne Chan took an old Chinese [7] idea for compounding a 3×3 magic square with itself to construct a

9×9 magic square, or a 3×3 with a 4×4 to make a pair of 12×12 magic squares, and devised a computer program [36] to extend this to very large squares. One of the squares is used as a frame and the other is incremented on placement in the appropriate position in the frame. Compounding preserves the row, column, pandiagonal, and regular [36] characteristics which are common to both squares, and even for the smallest case of ninth order there are very large numbers of distinct squares [37]. As a result we were able to set a new world record magic square size of $12{,}544 \times 12{,}544$. Since it is difficult to grasp a square of this size with numbers running from 1..157,351,936 we used a color scale to make an image which might pass for a piece of art [38]. In 2006 the world record for magic squares is still held by a smaller square of order 3001, because of rules which require writing or printing out the square on paper. We stopped at order $12{,}544 = 2^8 \times 7^2$ simply because it was the largest which we could write onto a CD. It is clear that larger magic squares could be stored on a DVD, or some higher capacity disks, but there is little point since potential applications will need fast access.

Let us note that Sudoku [39] is a special type of Latin square which bears some similarity to compound squares.

12. Multimagic squares

Christian Boyer [40, 41] has drawn attention to magic squares which remain magic when their elements are raised to integer powers (2 bimagic, 3 trimagic, etc.). An interesting mathematical analysis of multimagic squares has been given recently by Derksen, Eggermont, and van den Essen [42], which includes a contribution to compound squares. Rempel, Chan, and Loly have a related project underway [43].

13. A mechanical application: Moment of inertia

Another success growing out of teaching undergraduate classical mechanics for many years was the discovery of a new invariance, or universal property, for magic squares through calculations of their “moments of inertia” (essentially the inertia of the square through an axis perpendicular to its center), which eventually turned out to depend only on the order of the square, that is, the number of rows or columns [44]. The numbers in the magic square are replaced by corresponding multiples of a unit mass placed on a square unit lattice. In fact this was truly a “Eureka” moment for quite spontaneously I had the idea which fused long activity in teaching moment of inertia in introductory courses with a more recent activity in magic squares.

The moment of inertia I_n of a magic square of order n about an axis perpendicular to its center is obtained by summing mr^2 for each cell, where m is the number centered in a cell and r is the distance of the center of that cell from the center of the square measured in units of the nearest neighbor distance. For the Lo-shu square the corner cells then have their centers at a distance of $\sqrt{2}$ from the axis. We can now calculate the sum for the 3×3:

$$I_3 = [1+3+7+9](1)^2 + [2+4+6+8]\left(\sqrt{2}\right)^2 = 60. \tag{7}$$

The moments of inertia about the horizontal and vertical axes through the center are each 30, reminding us of the perpendicular axis theorem, which says that their sum gives the moment of inertia about the axis through the center and perpendicular to the plane.

When I used a data file for the complete 4×4 set (by courtesy of Harvey Heinz [45]) it was a surprise to find that they all gave $I_4 = 340$. I was then motivated to attempt a derivation, which was easy since the calculations only depended on the semi-magic property so that the parallel axis theorem and the perpendicular axis theorem could be used. In retrospect this could have been set as an examination problem in a sophomore course on classical mechanics:

$$I_n = \frac{1}{12} n^2 (n^4 - 1). \tag{8}$$

This remarkably simple formula recovers the results for $n = 3, 4$ above and is valid for arbitrary order. The derivation of the formula only depends on the row and column properties, and not on the diagonals of magic squares, so that it actually applies to the larger class of semi-magic squares which lack one or both diagonal magic sums of magic squares.

Of related interest, Abiyev *et al.* [46] have studied the center of mass of certain magic squares and suggest applications to robotics.

13.1 Row/column symmetry

In equation (8) one can factor out the magic linesum $\left(C_n\right)$:

$$I_n = \frac{1}{12} n^2 (n^4 - 1) = \frac{1}{6} n C_n \left(n^2 - 1\right) \tag{9}$$

for a result which applies to any square of equally spaced rows and columns with the same mass, C_n. The limit of a uniform continuous sheet agrees with the standard result using the calculus. It is clear that large random semi-magic squares tend towards the limiting value of this expression.

14. Magical cubes and hypercubes

Along with magic squares, there has long been an interest in magic cubes, going back at least to Leibniz [47], and then later in four-dimensional hypercubes following Riemann's n-dimensional geometry in the second half of the nineteenth century. There are now studies of higher dimensional hypercubes. There are some fascinating applications for magic cubes. We give a link to an image of a magic cube with spheres sized according to mass [48], as well as a paper [49] on perfect magic cubes, which also has an image.

Adam Rogers has recently helped extend my inertia ideas to the calculation of the full inertia tensor of magic cubes [50]. Row/column/pillar (RCP) symmetry means that any cube of equally spaced RCPs with the same mass in each will have the same form of inertia tensor. This work has also recently been reviewed by Ivars Peterson [51].

15. Electric quadrupoles

A new magic square topic has just emerged from my renewed involvement with our honors electromagnetism course.

The idea is to treat the numerical value of each element as an electric charge. It is soon clear for small squares that the dipole moment vanishes, so we then proceed to study the quadrupole moment. As a first thought I neutralized magic squares so that their elements ran from $-\left(n^2-1\right)/2$ to $+\left(n^2-1\right)/2$, but later Rogers and Loly [52] analyzed the multipole expansion for a normal magic square, finding that it takes care of many of the details. The full story involves calculating the quadrupole tensor, something beyond the scope of the present article. The short story is that the quadrupole tensor vanishes, so one then proceeds to the octupole!

16. Pandiagonal nonmagic squares (order $2^n = 2, 4, 8, 16, \ldots$)

Here the highlight is the discovery [53] of a small new class of purely pandiagonal nonmagic, number squares having dimensions of the powers of 2, with the additional property that the binary representation changes by just 1-bit in horizontal and vertical moves (they are counted from 0..15 in this example):

0	1	3	2
4	5	7	6
12	13	15	14
8	9	11	10

(10)

These squares derive from the Myers-Briggs dichotomous scheme of personality types. Then work with summer undergraduate Marcus Steeds [54], who devised a number of useful test tools, explored how generalizations of these squares are related to the Gray-code and square Karnaugh maps of digital logic design, a connection made by some of my third year engineering students a few years ago. I have also applied these ideas to ancient Chinese patterns based on the yin-yang duality [55].

Schindel [56] found that of the 3,465,216 fourth order pandiagonal nonmagic squares, 48 have the 1-bit property. This agrees with a symmetry argument made by Ian Cameron [57]. Meine and Schuett [58] of Siemens have suggested applications of these squares to cryptography and image processing [59], who also give a complementation transformation to magic squares.

17. From coupled oscillators to modern linear algebra

Undergraduate physics provides several examples in mechanics and wave motion (from coupled oscillators) where semi-magic matrices arise with algebraic or noninteger elements. The investigation of the mechanical problems had a vibrant interplay with mathematics for two centuries from the time of Huyghens and Newton. Huyghens, of course, is well known for his study of the isochrony of pendulum motion. An excellent chronology is found in Brillouin [60], whose study of waves in periodic systems is a *tour de force*.

A central mathematical theme in physical science and engineering concerns what are known as eigenvalue problems. These involve homogeneous linear equations which only have nontrivial solutions if the determinant of the coefficients vanishes, with as many solutions as the number of equations. These issues can be clarified by using a specific example for which the coupled oscillator is ideal. At the same time we can prepare the ground for studying magic square matrices in their own right.

17.1 Homogeneous simultaneous equations for the coupled oscillator

In the usual description of this one-dimensional problem [61] the equations of motion for masses M displaced along the x-direction from their equilibrium positions by x_1 and x_2, a coupling spring of force constant γ, and with each tied to fixed posts at opposite ends by springs of force constant κ are:

$$M\ddot{x}_1 + (\kappa + \gamma)x_1 - \gamma x_2 = 0 \tag{11}$$

$$M\ddot{x}_2 + (\kappa + \gamma)x_2 - \gamma x_1 = 0. \tag{12}$$

These are simplified by taking out a simple time-dependence: $x(t) = B\exp(i\omega t)$ for:

$$(\kappa + \gamma - M\omega^2)B_1 - \gamma B_2 = 0 \tag{13}$$

$$-\gamma B_1 + (\kappa + \gamma - M\omega^2)B_2 = 0. \tag{14}$$

Instead of the general approach of setting the determinant of the coefficients of these simultaneous equations to zero, this problem may be solved simply by forming ratios of the variables:

$$\frac{B_1}{B_2} = \frac{\gamma}{(\kappa+\gamma) - M\omega^2} = \frac{(\kappa+\gamma) - M\,\omega^2}{\gamma}. \tag{15}$$

Cross multiplication of the second equality results in a quadratic equation in ω^2, with two solutions, one, $\omega^2 = \kappa/M$, is just the frequency of each oscillator without coupling, and the other higher, $\omega^2 = (\kappa + 2\gamma)/M$. Extended to a chain of two or more alternating masses and springs, we have the origin of the gaps in the spectrum which are a characteristic feature of solid state physics. We continue this example after introducing some essential matrix operations, which are easy enough to cover in high school.

18. Determinants and matrices

Cayley initiated matrix theory in 1846, followed by contributions from Peirce, Hamilton, Poincaré, and Sylvester. We highlight the issues of interest with respect to magic squares, the semi-magic matrices and tensors arising in mechanics, and the related interest for pandiagonal nonmagic squares with a brief discussion using 2×2 matrices.

18.1 Matrix multiplication

Consider the usual matrix-vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} aP + bQ \\ cP + dQ \end{bmatrix}. \tag{16}$$

Clearly if $P = Q = 1$, this sums the rows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix}. \tag{17}$$

This $[1, 1]$ vector will be referred to as a diagonal (corner to opposite corner or, 2-agonal) vector, and generalizes to higher orders. However if one takes the original matrix operator to act from the right onto a row vector on the left, as a "left-hand" problem, then one finds the column sums of the original matrix:

$$\begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Pa + Qc & Pb + Qd \end{bmatrix} \tag{18}$$

if $P = Q = 1$, this sums the columns:

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \end{bmatrix}. \tag{19}$$

This may also be achieved by multiplication of the transposed matrix on the left with a column on the right.

18.2 Eigenproblems: Eigenvectors and eigenvalues

My interest in these issues began with an observation from experimental computing of a few different order magic squares with *Mathematica* during a sophomore course which I was teaching. My colleague Frank Hruska had published a relevant and stimulating paper [62], David Lavis drew my attention to left and right eigenvectors, whilst another, Joe Williams, knew from teaching linear algebra that this eigenvector adds row elements. The utility of the "n-agonal" eigenvector $[1,1,1,\ldots]$ of the n-cube is seen by showing how the rows sum in the Lo-shu magic square:

$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{20}$$

where we have factored out the eigenvalue 15 to show the action of the matrix operator in leaving the eigenvector unchanged. The columns have the same eigenvalue as follows from the transposed matrix:

$$\begin{bmatrix} 3 & 4 & 8 \\ 5 & 9 & 1 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{21}$$

Alternatively we can work with a row eigenvector on the left with the original matrix:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left[\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix}\right] = 15 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}. \tag{22}$$

We must note that the n-agonal eigenvector is both a left and a right eigenvector, and that this property depends only on the semi-magic property. An immediate application is now afforded by the coupled oscillator.

19. Coupled oscillator eigenvectors

When written in modern matrix notation, the semi-magic nature of the characteristic equations (13) and (14) is apparent:

$$\begin{bmatrix} \kappa+\gamma & -\gamma \\ -\gamma & \kappa+\gamma \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M\omega^2 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{23}$$

It is immediately clear that a solution is the $[1,1]$ eigenvector, both as a right, and as a left eigenvector. It has the lower eigenvalue of $\omega^2 = \kappa/M$. The other eigenvector is $[1,-1]$, which corresponds to the higher solution of $\omega^2 = (\kappa + 2\gamma)/M$.

A similar semi-magic property is also found for the full moment of inertia tensor of magic cubes [50].

19.1 Left and right eigenvectors

Of significant interest for our studies of magic squares, this topic is important for teaching linear algebra beyond the introductory course. The use of magic square examples already occurs in such courses, but it can be used even more seriously. The existence of identical left and right eigenvectors implies deeper properties giving rise to the theorem of biorthogonality, and the theorem of Perron [63]. These theorems develop deep links between the left and right eigenvectors and eigenvalues. As such, magic squares are insightful examples for advanced linear algebra courses.

Software such as *Mathematica*, Maple, and MATLAB can be profitably employed in such studies. Indeed, MATLAB [64] has initiated some of this already by including a function *magic*(n) which returns a magic square from one of three algorithms, one each for odd, even, and doubly-even cases. A drawback with *magic*(n) is that the single squares which result are not representative of the richness of the spectrum of magic squares of a given order, save for $n > 3$.

19.2 Eigenvalues of magical squares

We have nearly finished a study of the eigenvalues of magic square matrices [65]. For the 880 distinct 4×4 cases in the 12 Dudeney groups, we find that members of the first six (singular) groups have three distinct eigenvalue patterns, with a subset of the first three groups having three zero eigenvalues, while the last six (nonsingular) groups have two further eigenvalue patterns. Also if the 1-bit pandiagonal nonmagic squares discussed earlier are treated as matrices they possess examples with just two nonzero eigenvalues for any order [54].

I can add here that eighth order Franklin squares have three nonzero eigenvalues, as do also the corresponding most-perfect squares.

20. Conclusion

Further information on the history of magic squares may be found in recent books by Frank Swetz [7], a mathematics educator, by René Descombes [66], as well as Clifford Pickover [9]. Those sources also enlarge on the philosophical aspects, which began in China as a cosmology, or organizing scheme.

There are opportunities to enrich teaching in classical physics, and likely in quantum physics as well. Certainly more can be done in the context of teaching linear algebra, which can begin in high school. I have found wonderful opportunities for students to cooperate in some group work as summer research assistants, indeed their enthusiasm and initiative in tackling problems has been gratifying.

While our recent count of Franklin squares was limited to eighth order, we outlined some further issues [1] which we hope to see addressed in the future.

Acknowledgments

This article is an elaboration of a poster talk given at NKS 2006, which has been augmented as a result of feedback from Stephen Wolfram and others. Ian Cameron, David Lavis, Frank Hruska, Joe Williams, Walter Trump, John Hendricks, and Harvey Heinz have helped with advice on various aspects of this study, which engaged the following undergraduates: Marcus Steeds, Wayne Chan, Adam Rogers, Matt Rempel, and Dan Schindel.

Critical funding for this research was provided in 2003 by the Winnipeg Foundation Post Secondary Grants Program, especially for support to Adam Rogers and Matt Rempel. My colleagues Professors John Vail and Tom Osborn generously let their summer student Daniel Schindel work part time on some of these projects during 2003 and 2004.

Stephen Wolfram posed several questions at my poster, one concerning the line path which can be drawn through successive numbers of a Franklin square. Stephen had in mind an underlying group theory. With the advent of larger complete sets of magic squares more study in this direction would be timely. Stephen was also interested in the large nullspaces of some of our magic squares, and in what one might conclude about random matrices.

References

[1] D. Schindel, M. Rempel, and P. Loly, "Enumerating the Bent Diagonal Squares of Dr. Benjamin Franklin FRS," *Proceedings of the Royal Society A: Physical, Mathematical and Engineering*, **462** (2006) 2271–2279.

[2] P. C. Pasles, "The Lost Squares of Dr. Franklin: Ben Franklin's Missing Squares and the Secret of the Magic Circle," *American Mathematical Monthly*, **108** (2001) 489–511.

[3] M. M. Ahmed, "How Many Squares Are There, Mr. Franklin?: Constructing and Enumerating Franklin Squares," *American Mathematical Monthly*, **111** (2004) 394–410.

[4] M. M. Ahmed, *Algebraic Combinatorics of Magic Squares*, Ph.D. dissertation, University of California, Davis, 2004.

[5] I. Peterson, "Counting Franklin's Magic Squares," *Science News Online* (24 June 2006); www.sciencenews.org/articles/20060624/mathtrek.asp.

[6] K. Ollerenshaw and D. S. Brée, *Most-perfect Pandiagonal Magic Squares: Their Construction and Enumeration* (The Institute of Mathematics and its Applications, Southend-on-Sea, U.K., 1998).

[7] F. J. Swetz, *Legacy of the Luoshu: The 4000 Year Search for the Meaning of the Magic Square of Order Three* (Open Court, Chicago, 2002).

[8] C. Bragdon, *The Frozen Fountain: Being Essays on Architecture and the Art of Design in Space* (Alfred A. Knopf, New York, 1932); Kessinger Publishing's Rare Mystical Reprints, www.kessinger.net.

[9] C. Pickover, *The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures Across Dimensions* (Princeton University Press, 2002).

[10] M. Frénicle de Bessy, "Des Quarrez ou Tables Magiques," *Histoire de l'Académie royale des sciences avec les mémoires de mathématique et de physique tirés des registres de cette Académie*, **4** (1729) 209–304; gallica.bnf.fr/ark:/12148/bpt6k34948.

[11] M. Frénicle de Bessy, "Table Generale des Quarrez Magiques de quatre," *Histoire de l'Académie royale des sciences avec les mémoires de mathématique et de physique tirés des registres de cette Académie*, **4** (1729) 305–354; gallica.bnf.fr/ark:/12148/bpt6k34948.

[12] I. B. Myers with P. B. Myers, *Gifts Differing: Understanding Personality Type* (CPP Books, Palo Alto, CA, 1980, 1993).

[13] M. L. von Franz, *Number and Time: Reflections Leading Toward a Unification of Depth Psychology and Physics* (Northwestern University Press, 1974).

[14] E. McClintock, "On the Most Perfect Forms of Magic Squares, with Methods for Their Production," *American Journal of Mathematics*, **19** (1897) 99–120.

[15] K. Ollerenshaw and D. S. Brée, *Transposition Cipher*, WO 99/66671 Patent (17 June 1999) 1–33.

[16] Ph. W. Besslich, "Comments on Electronic Techniques for Pictorial Image Reproduction," *IEEE Transactions on Communications*, **31** (1983) 846–847.

[17] Ph. W. Besslich, "A Method for the Generation and Processing of Dyadic Indexed Data," *IEEE Transactions on Computers*, **C-32**(5) (1983) 487–494.

[18] R. Schroeppel, (1973), see M. Gardner, "Mathematical Games," *Scientific American*, **249** (Jan. 1976) 118.

[19] K. Pinn and C. Wieczerkowski, "Number of Magic Squares from Parallel Tempering Monte Carlo," *International Journal of Modern Physics C*, **9**(4) (1998) 541–546.

[20] W. Trump, *Notes on Magic Squares and Cubes*, (2003); www.trump.de/magic-squares.

[21] H. B. Meyer, *Magic Squares of Order 4*, (2005); www.faust.fr.bw.schule.de/mhb/backtrack/mag4en.htm.

[22] P. D. Loly and D. G. Schindel, "A Simplified Demonstration of Counting the 880 Fourth Order Magic Squares Using *Mathematica*," NKS 2006 Wolfram Science Conference, (2006); www.wolframscience.com/conference/2006/presentations/materials/loly.nb.

[23] Eric W. Weisstein, "Magic Square," from *MathWorld*–A Wolfram Web Resource, (2003); mathworld.wolfram.com/MagicSquare.html.

[24] D. Mackenzie, "Quadramagicology," *New Scientist,* **180** (2003) 50–53.

[25] M. M. Ahmed, J. De Loera, and R. Hemmeke, "Polyhedral Cones of Magic Cubes and Squares," in *New Directions in Computational Geometry, The Goodman-Pollack Festschrift Volume*, edited by Aronov *et al.* (Springer-Verlag, New York, 2003).

[26] M. Beck, M. Cohen, J. Cuomo, and P. Gribelyuk, "The Number of 'Magic' Squares and Hypercubes," *American Mathematical Monthly*, **110** (2003) 707–717.

[27] Joint AMS-IMS-SIAM Summer Research Conference on Integer Points in Polyhedra, Geometry, Number Theory, Algebra, Optimization, 2003; proceedings to appear in *Contemporary Mathematics* (Proceedings of the Summer Research Conference on Integer Points in Polyhedra, July 13–17, 2003 in Snowbird, Utah); ams.org/meetings/src-barvin.html.

[28] M. Beck, "Magic Squares and the Birkhoff Polytope," in *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra* (Banff International Research Station Summer School, 2005; pims.math.ca/birs/workshops/2005/05ss027.

[29] M. Amela, *Structured* 8×8 *Franklin Squares*, (2006); www.region.com.ar/amela/franklinsquares.

[30] Yu. V. Chebrakov, "Analytical Formulae and Algorithms for Constructing Magic Squares from an Arbitrary Set of 16 Numbers," *Smarandache Notions Journal*, **8** (1997) 28–45; www.gallup.unm.edu/~smarandache.

[31] C. Jacobs and H. Hadler, *Evolution Strategies*, (1998); www.dcs.napier.ac.uk/~benp/summerschool/summerschool.htm.

[32] I. McDonald, "NuSBDS: An Easy to Use Symmetry Breaking System," in *Lecture Notes in Computer Science*, Volume 2833 (2003) 985 (Springer, Berlin and Heidelberg, ISSN: 0302-9743).

[33] B. Hnich, and T. Walsh, "Why Channel? Multiple Viewpoints for Branching Heuristics," *Proceedings of the CP03 Second International Workshop on Modelling and Reformulating Constraint Satisfaction Problems: Towards Systematisation and Automation (2003)*, in *Principles and Practice*

of Constraint Programming—CP 2003: Ninth International Conference, CP 2003, edited by Francesca Rossi (Kinsale, Ireland, September 29–October 3, 2003, ISBN: 3-540-20202-1).

[34] C. M. Roney-Dougal, I. P. Gent, T. Kelsey, and S. Linton, "Tractable Symmetry Breaking Using Restricted Search Trees," *Proceedings of the Sixteenth European Conference on Artificial Intelligence (ECAI 2004)*, Valencia, Spain.

[35] T. Xie and L. Kang, "An Evolutionary Algorithm for Magic Squares," The 2003 Congress on Evolutionary Computation, CEC'03, *Evolutionary Computation (IEEE)*, **2** (2003) 8–12.

[36] W. Chan and P. D. Loly, "Iterative Compounding of Square Matrices to Generate Large-Order Magic Squares," *Mathematics Today (IMA)*, **38**(4) (2002) 113–118.

[37] J. Bellew, "Counting the Number of Compound and Nasik Magic Squares," *Mathematics Today*, August (1997) 111–118.

[38] W. S. Chan and P. D. Loly, *Colour Image of 12,544th Order Compound Magic Square*, (2007); home.cc.umanitoba.ca/~loly/smallmath.jpg.

[39] Ed Pegg Jr. and Eric W. Weisstein, "Sudoku," from *MathWorld*–A Wolfram Web Resource, (2006); mathworld.wolfram.com/Sudoku.html.

[40] C. Boyer, *Multimagic Squares*, (2006); www.multimagie.com.

[41] C. Boyer, "Some Notes on the Magic Squares of Squares Problem," *Mathematical Intellingencer*, **27**(2) (2005) 52–64.

[42] H. Derksen, C. Eggermont, and A. van den Essen, *Multimagic Squares*, arXiv:math.CO/0504083 v2 5 (Apr 2005), to appear in *American Mathematical Monthly*.

[43] M. Rempel, W. S. Chan, and P. D. Loly, "Compounding Regular and Multimagic Squares, and a Second Method," (in preparation, 2007).

[44] P. D. Loly, "The Invariance of the Moment of Inertia of Magic Squares," *The Mathematical Gazette*, **88** (511) (2004) 151–153.

[45] H. Heinz, *Order-4 Magic Squares*, (2006); www.geocities.com/~harveyh/order4list.htm.

[46] A-A. Abiyev, *et al.*, "Investigation of Center of Mass by Using Magic Squares and Its Possible Engineering Applications," *Robotics and Autonomous Systems*, **49** (2004) 219–226.

[47] H. Heinz, *Magic Cubes—Introduction*, (2006); members.shaw.ca/hdhcubes.

[48] P. D. Loly, *Magic Cube with Massive Balls Using Visual Python*, (2005); home.cc.umanitoba.ca/~loly/squareBalls6.bmp.

[49] C. Boyer, *Perfect Magic Cubes*, (2006); cboyer.club.fr/multimagie//English/Perfectcubes.htm.

[50] A. Rogers and P. D. Loly, "The Inertia Tensor of Magic Cubes," *American Journal of Physics*, **72**(6) (2004) 786–789.

[51] I. Peterson, "Magic Square Physics," *Science News Online*, (1 July 2006); www.sciencenews.org/articles/20060701/mathtrek.asp.

[52] A. Rogers and P. D. Loly, "The Electric Multipole Expansion for a Magic Cube," *European Journal of Physics*, **26** (2005) 809–813.

[53] P. D. Loly, "A Purely Pandiagonal 4×4 Square and the Myers-Briggs Type Table," *Journal of Recreational Mathematics*, **31** (1) (2002–2003) 29–31.

[54] P. D. Loly and M. Steeds, "A New Class of Pandiagonal Non-Magic Squares," *International Journal of Mathematics Education Science Technology*, **36**(4) (2005) 375–388.

[55] P. D. Loly, "A Logical Way of Ordering the Trigrams and Hexagrams of the Yijing," *The Oracle: The Journal of Yijing Studies*, **2**(12) (2002) 2–13.

[56] D. G. Schindel, private communication, 2003.

[57] I. Cameron, private communication, 2003.

[58] S. Meine and D. Schuett, "On Karnaugh Maps and Magic Squares," *Informatik-Spektrum*, **28**(2) (2005) 120–123.

[59] D. Schuett, U. Eckhardt, and P. Suda, "Boot Algebras," *TR-92-039* (International Computer Science Institute, Berkeley, CA, 1992).

[60] L. Brillouin, *Wave Propagation in Periodic Structures* (McGraw-Hill, New York, 1946).

[61] J. B. Marion and R. K. Thornton, *Classical Dynamics of Particles and Systems*, *Fourth Edition* (Harcourt, Brace, Jovanovich, New York, 1995).

[62] F. Hruska, "Magic Squares, Matrices, Planes and Angles," *Journal of Recreational Mathematics*, **23**(3) (1991) 183–189.

[63] R. B. Mattingly, "Even Order Regular Magic Squares are Singular," *American Mathematical Monthly*, **107** (2000) 777.

[64] C. Moler, "MATLAB's Magical Mystery Tour," *The MathWorks Newsletter*, **7**(1) (1993) 8–9.

[65] P. D. Loly, D. G. Schindel, W. Trump, and I. D. Cameron, "Magic Square Eigenvalues and Characteristic Equations," (in preparation, 2007).

[66] R. Descombes, *Les Carrés Magiques, Histoire, théorie et technique du carré magique, de l'Antiquité aux recherches actuelles* (Vuibert, Paris, 2001).