A Note on the Game of Life in Hexagonal and Pentagonal Tessellations

Carter Bays
Department of Computer Science and Engineering,
University of South Carolina,
Columbia, SC

1. Introduction

Most readers are familiar with cellular automata (CA) utilizing squares as cells, and are also familiar with the most famous automaton, Conway's Game of Life. This game is played on an infinite grid of squares, and has been thoroughly explored in many publications [1, 2] as well as on numerous websites.

In 1994 in this journal the author introduced several games of life in the triangular tessellation [3]. It now appears, not surprisingly, that other tessellations also support games of life.

1.1 Some terminology

We define an *oscillator* as a finite shape with an inert nonliving boundary and a finite period ≥1. A boundary is *inert* if we can place the oscillator, boundary and all, in a universe of nonliving cells without altering the behavior of the oscillator (see Figure 1). In many cases, a period >1 oscillator may, during its period, exhibit phases where the shape may break into two (or more) parts, each of which appears to have a boundary. Naturally, in this case the boundary is not inert and the two parts would usually not constitute two oscillators.

An oscillator that translates is called a *glider*.

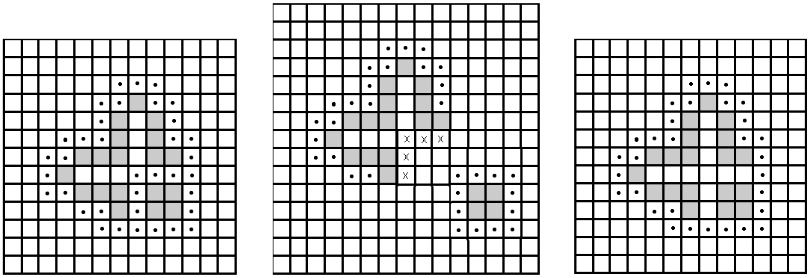
A rule is a game of life (GL) rule if it satisfies the criteria below.

A. When counting the neighbors of a cell, all touching neighbors are considered and treated the same.

B. At least one glider exists.

C. Start with a finite wrapped universe that is completely filled with a random pattern. Then after a finite number of generations, all such patterns eventually must either disappear, or decompose into one or more oscillators. Rules exhibiting this property are said to be *stable*.

Condition C requires that random patterns do not grow without bounds. In fact, random patterns under most rules either quickly stabilize by decomposing into several small oscillators, or exhibit unbounded growth



(a) (b) (c)

Figure 1. Two potential 2,3/3 oscillators (period = 1) are shown in (a) with the boundaries indicated by the dotted cells. If we separate the two forms (b) we find that the lower form is indeed an oscillator, but the upper form changes its behavior and starts to disintegrate. This change initiates along the boundary portion indicated by "x", which hence is not an inert boundary and the upper shape alone is not an oscillator. But in (c) we note that the boundary is inert, hence the form within this boundary is an oscillator.

and churn on randomly forever. The GL rule for Conway's Life is a borderline case, and that contributes to its richness. Moreover it is possible to construct quadratic growth patterns under Conway's Life rather easily. But Condition C eliminates the possibility that they could persist. Intuitively, all we are saying is that under a GL rule, random patterns eventually settle down into zero or more elementary forms.

In [4] this author introduced the notation E_l, E_u, F_l, F_u for specifying GL rules. Here, E_l, E_u (the "environment" rule) gives the range of live touching neighbors required so that a currently live cell remains alive the next generation. F_l, F_u (the "fertility" rule) specifies the range for live touching neighbors so that a currently dead cell will come to life in the next generation. Thus, Conway's Life is given by the rule 2,3,3,3, which says, "a dead cell comes to life if it is touching exactly three live neighbors, and a live cell will remain alive if it is touching two or three live neighbors." (Note that in the square grid, each cell, or square, has exactly eight neighbors.)

Unfortunately the notation from [4] is somewhat limited. A better and more frequently used notation is now given. Here we do not specify a range; rather we give individual values, $E_1, E_2, \ldots / F_1, F_2, \ldots$ in ascending order, for environment and fertility. Hence Conway's Life is specified by 2,3/3. The rule 2,5/3,6 (not a GL rule) means, "a dead cell comes to life if it is touching exactly three or six live neighbors, and a live cell will remain alive if it is touching two or five live neighbors." This notation, used throughout this paper, expands the number of rules that can be specified; indeed it turns out that rule 2,4,5/3 is also a GL rule, exhibiting the glider shown in Figure 2. One might note that a

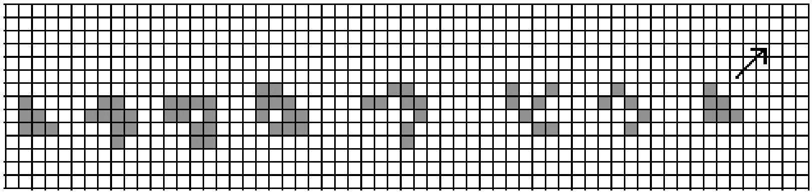


Figure 2. The 2,4,5/3 glider has a period of seven and moves one cell in the direction shown.

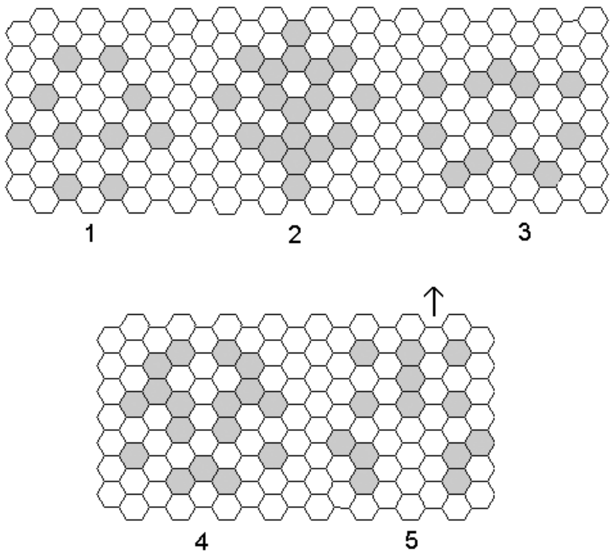


Figure 3. The 3,5/2 hex glider has a period of five, moving one cell vertically (up).

popular rule commonly known as "3-4 Life," (the rule 3,4/3,4) is not a true GL rule, as random blobs exhibit unbounded growth.

2. Hexagonal games of life

The hexagonal tessellation has been investigated extensively in the past, but no valid game of life rule has turned up. The closest rules have involved utilizing "Golay Surrounds" [6], which treat neighbors differently depending upon their relative position around the cell in question. In that case, a couple of gliders were discovered in the hexagonal grid [6], but these rules do not qualify as GL rules (see Condition A).

After considerable searching, a valid hexagonal GL rule has been found. The rule 3,5/2 satisfies all three criteria; its discovered glider has a period of five and is illustrated in Figure 3. Of course, this rule is not

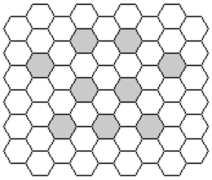


Figure 4. The 3/2,4,5 hex glider has a period of 10 and moves vertically two cells. Unfortunately, the rule 3/2,4,5 does not quite qualify as a GL rule.

nearly as rich as Conway's Life; however there may be other gliders, as well as interesting patterns and oscillators, but those remain for future work.

Another rule, 3/2,4,5, supports a period 10 glider (Figure 4); unfortunately, this rule does not qualify (barely) as a GL rule because sufficiently large random blobs exhibit instability by growing slowly, churning on forever.

Most rules can be eliminated as GL candidates if we note that F_1 must be exactly 2, for if $F_1 = 1$, then the rule will be unstable, and if $F_1 = 3$, then no glider is possible.

3. The pentagonal tessellation

There are many ways to tessellate with pentagons (see Figure 5). Here we have chosen the "Cairo tiling," which in its most regular form is composed of equilateral "isosceles" pentagons. It is of interest in that the neighbor count (seven) is between that for the hexagonal and square tessellations (six and eight respectively).

So far, one GL rule has been discovered, namely 2,3/3,4,6. This rule supports an unusual glider with a period of 48 (Figure 6). There may be other GL rules, but their gliders will be difficult to find, since symmetric random arrangements are not as common in this tessellation, and such arrangements greatly speed up the discovery process [7]. Nevertheless there are probably other pentagonal GL rules, given the variety of topologically distinct tessellations.

4. The triangular tessellation

This grid was investigated in [3]; since then, more GL rules have been discovered (Figure 7). The glider for 2,7/3 is of interest in that during several of its states, it only contains four cells. It has a period of 18 and at several points looms rather large, looking somewhat like a fish as it wanders torturously on its way. It is the most commonly occurring object under this rule when starting with grids composed of random live cells. Other recently discovered GL rules are 2/3, 3,5/4, 2,4/4,6, and

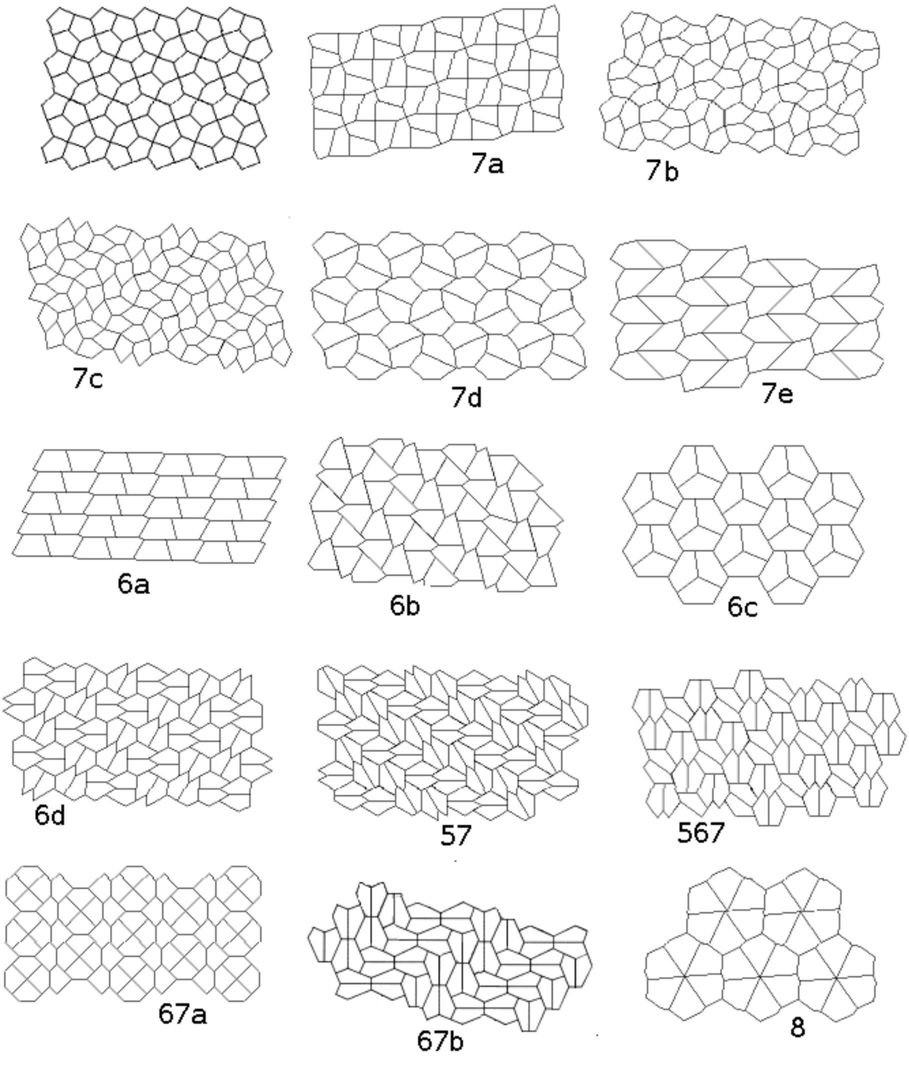


Figure 5. 14 tiling types for convex pentagons have been cataloged [8]. These are depicted here and have been arranged to reflect the number of touching neighbors for each cell. In the case where the neighbor count varies depending upon the cell, all the different counts are given. For example "67" means that some cells have 6 neighbors, others 7. Here there are two tilings reflecting these counts(67a and 67b). The Cairo tiling is at the upper left and is topologically equivalent to 7a and 7b. Note also that 7c and 7d are topologically equivalent. All other arrangements are distinct.

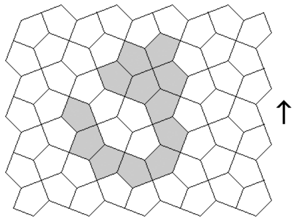


Figure 6. The pentagonal glider 2,3/3,4,6 has a period of 48 and moves six units vertically (up) for each cycle.

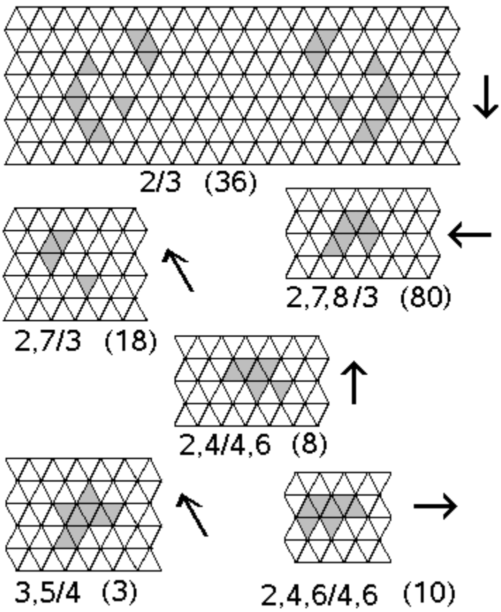


Figure 7. Other triangular GL rules. The incredible 2/3 glider has a period of 36 and at the end of the cycle will have moved four units vertically (down). The 2,7/3 glider exhibits several states containing only four or five cells. It has a period of 18, after which it will have moved one cell in the direction shown. It is the most common oscillator for this rule, which unfortunately exhibits a paucity of other simple oscillators. An even more remarkable GL rule is 2,7,8/3. The illustrated glider has a period of 80, moving 12 cells to the left. This glider is quite remarkable, spewing off much material as it moves along (at one point, there are more than 50 live cells in the configuration). 2,7,8/3 also supports the glider for rule 2,7/3. The 2,4/4,6 glider has a period of eight. This orientation moves two units up per period. The 3,5/4 glider has a period of three and moves one unit to the upper left as shown. The 2,4,6/4,6 glider has a period of 10 and moves one cell to the right. It slightly resembles the 2,4/4,6 glider, but note that its movement is parallel to the bases of the triangles instead of perpendicular.

2,4,6/4,6. The 2/3 glider is rather remarkable, being huge and having a period of 36. These five additional GL rules brings the total number of triangular GL rules to 11. More will probably surface.

5. Further work

Most of the results for this paper were found using the applets referred to on my website [9] (under "cellular automata"). These applets allow one to easily test rules for stability with random configurations, search for gliders, and so forth, by performing experiments with small random blobs. The ability to impose symmetry on these blobs greatly increases the probability of finding interesting shapes.

It should be emphasized that in spite of all these game of life rules in various tessellations, none has yet been found that rivals the richness of Conway's Game of Life. Only time and much investigation will reveal if this fact remains true.

Acknowledgments

The following students worked on the program(s) that were ultimately very helpful in contributing to the research for this paper: Darren Braynard, Tom Johnson, Paras Kinariwala, Martin Dessart, Nathan Childs, Bryan Richardson, Jason Cromer, Jason Areheart, Richard Smith, Kenny Frazier, Eric Young, William Biggerstaff, Nancy Rettew, Winnie Wang, and especially Joel Froese.

I also wish to acknowledge the very useful remarks made by the two reviewers of this paper and the editor of this journal, Todd Rowland.

References

[1] Martin Gardner, "The Fantastic Combinations of John Conway's New Solitaire Game 'Life,'" *Scientific American*, **223** (October 1970) 120–123.

[2] William Poundstone, *The Recursive Universe* (William Morrow & Co., New York, 1984).

[3] Carter Bays, "Cellular Automata in the Triangular Tessellation," *Complex Systems*, **8** (1994) 127–150.

[4] Carter Bays, "Candidates for the Game of Life in Three Dimensions," *Complex Systems*, **1** (1987) 373–400.

[5] Stephen Wolfram, *A New Kind of Science* (Wolfram Media, Inc., Champaign, IL, 2002).

[6] Kendall Preston, *Modern Cellular Automata Theory and Applications* (Plenum Press, New York, 1984).

[7] Carter Bays, "Further Notes on the Game of Three Dimensional Life," *Complex Systems*, **8** (1994) 67–73.

[8] Teruhisa Sugimoto and Tohru Ogawa, "Tiling Problem of Convex Pentagons," *Forma*, **15** (2000) 75–79.

[9] Carter Bays' Home Page currently resides at www.cse.sc.edu/~bays.