The Relationship between One-dimensional Continuous Cellular Automata and One-dimensional Nonlinear Dynamical Systems

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In this paper the relationship between systems of real-valued, or continuous, one-dimensional cellular automata (CA) and a corresponding one-dimensional nonlinear dynamical system is investigated. The CA will be defined in a periodic domain and will satisfy requirements on the function used to assign new values; the corresponding one-dimensional dynamical system will be derived from the function used to assign new values. It is shown that the qualitative behavior of the CA—the existence of a stable orbit with the same value at all cells—is determined by the corresponding one-dimensional system and the geometry of the CA.

1. Analysis

One-dimensional cellular automata (CA) are a class of dynamical systems, discrete in time and space. The discretized space is a regular uniform lattice which in one dimension becomes a one-dimensional array of "cells," at each cell there is a variable. The values at each time step are determined by the values of the variables at the previous time step which are within some neighborhood of the variable [6].

Typically the values assigned to the cells are chosen from a finite set. Here we will take the values from the unit interval [0,1] giving real-valued or continuous cellular automata (CCA). In [5] Pederson uses real-valued functions that interpolate between the values of a discrete function, analyzing how the approximation of a discrete system compares to the discrete system. In [4] Ostrov and Rucker use CCA to study nonlinear wave equations.

In this paper we investigate a system where the lattice is periodic in space and the variable is real-valued and restricted to values in [0, 1]. That is, there are N cells, $\{x_1, \ldots, x_N\}$ with each x_i adjacent to x_{i-1} and x_{i+1} and x_1 is adjacent to x_N, and at each of these cells there is a real value in [0, 1].

We shall also require that the rule used to assign new values is a symmetric function, $f_\mu : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow [0, 1]$, with

$$f_\mu(x_1, x_2, x_3) = f_\mu(x_i, x_j, x_k), \tag{1}$$

where (i, j, k) is any permution of $\{1, 2, 3\}$ and $f_\mu(x, x, x)$ has only one attractive orbit under iteration.

We will use linear stability analysis to determine a condition when a system of this type has a stable orbit determined by the associated one-dimensional dynamical system.

Let us start by letting the $\vec{u}^n = (u_1^n, u_2^n, \ldots, u_N^n)$ be the values at each location at time step n. Then our values will satisfy

$$u_i^{n+1} = f_\mu(u_{i-1}^n, u_i^n, u_{i+1}^n). \tag{2}$$

We can introduce the average value

$$\bar{u}^n = \frac{\sum_{i=1}^N u_i^n}{N} \tag{3}$$

and the difference

$$\epsilon_i^n = u_i^n - \bar{u}^n. \tag{4}$$

Linearizing will give

$$
\begin{aligned}
\bar{u}^{n+1} + \epsilon_i^{n+1} &= f_\mu(\bar{u}^n + \epsilon_{i-1}^n, \bar{u}^n + \epsilon_i^n, \bar{u}^n + \epsilon_i^n) \\
&= f_\mu(\bar{u}^n, \bar{u}^n, \bar{u}^n) \\
&\quad + \frac{\partial f_\mu}{\partial u_{i-1}}(\bar{u}^n, \bar{u}^n, \bar{u}^n)\epsilon_{i-1}^n + \frac{\partial f_\mu}{\partial u_i}(\bar{u}^n, \bar{u}^n, \bar{u}^n)\epsilon_i^n \\
&\quad + \frac{\partial f_\mu}{\partial u_{i+1}}(\bar{u}^n, \bar{u}^n, \bar{u}^n)\epsilon_{i+1}^n + O(\max_i\{(\epsilon_i^n)^2\}).
\end{aligned}
\tag{5}
$$

Exploiting symmetry will let us define

$$\bar{f}_\mu(\bar{u}^n) = f_\mu(\bar{u}^n, \bar{u}^n, \bar{u}^n) \tag{6}$$

and

$$
\begin{aligned}
\frac{\partial \bar{f}_\mu}{\partial u}(\bar{u}^n) &= \frac{\partial f_\mu}{\partial u_{i-1}}(\bar{u}^n, \bar{u}^n, \bar{u}^n) \\
&= \frac{\partial f_\mu}{\partial u_i}(\bar{u}^n, \bar{u}^n, \bar{u}^n) \\
&= \frac{\partial f_\mu}{\partial u_{i+1}}(\bar{u}^n, \bar{u}^n, \bar{u}^n).
\end{aligned}
\tag{7}
$$

Then we will have

$$
\begin{aligned}
\bar{u}^{n+1} + \epsilon_i^{n+1} &= f_\mu(\bar{u}^n + \epsilon_{i-1}^n, \bar{u}^n + \epsilon_i^n, \bar{u}^n + \epsilon_i^n) \\
&= \bar{f}_\mu(\bar{u}^n) + \frac{\partial \bar{f}_\mu}{\partial u}(\bar{u}^n, \bar{u}^n, \bar{u}^n)(\epsilon_{i-1}^n + \epsilon_i^n + \epsilon_{i+1}^n) \\
&\quad + O(\max_i\{(\epsilon_i^n)^2\}).
\end{aligned}
\tag{8}
$$

Summing each of these expressions from $i = 1$ to $i = N$ and dividing by N will give

$$\bar{u}^{n+1} = \bar{f}_\mu(\bar{u}^n) + O(\max_i\{(\epsilon_i^n)^2\}) \tag{9}$$

because $\sum_{i=1}^{N} \epsilon_i^n = 0$. We can then subtract this expression from equation (7) to get for each ϵ_i^{n+1}

$$\epsilon_i^{n+1} = \frac{\partial \bar{f}_\mu}{\partial u}(\bar{u}^n)(\epsilon_{i-1}^n + \epsilon_i^n + \epsilon_{i+1}^n) + O(\max_i\{(\epsilon_i^n)^2\}). \tag{10}$$

If we let

$$
B = \begin{bmatrix}
1 & 1 & 0 & & \cdots & & 0 & 1 \\
1 & 1 & 1 & 0 & \cdots & & & 0 \\
0 & 1 & 1 & 1 & 0 & \cdots & & \vdots \\
\vdots & & & & \ddots & & & 0 \\
0 & & \cdots & & 0 & 1 & 1 & 1 \\
1 & 0 & & \cdots & & 0 & 1 & 1
\end{bmatrix}, \tag{11}
$$

(i.e., an $N \times N$ matrix with 1s along the diagonal, the subdiagonal, and the superdiagonal as well as in the upper right and lower left corners, and 0s for all other entries) and

$$\vec{\epsilon}^{\,n} = (\epsilon_1^n, \epsilon_2^n, ..., \epsilon_N^n)^T \tag{12}$$

then we can write

$$\vec{\epsilon}^{\,n+1} = \frac{\partial \bar{f}_\mu}{\partial u}(\bar{u}^n)B\vec{\epsilon}^{\,n} + O(\max_i\{(\epsilon_i^n)^2\}). \tag{13}$$

The matrix B is a *circulant matrix*–each row is equal to the previous row shifted one position to the right modulo the size of the matrix. This special structure tells us the eigenvalues and eigenvectors. The eigenvalues are given by

$$\lambda_j = a_1 + a_2 r_j + a_3 r_j^2 + \cdots + a_n r_j^{n-1} \tag{14}$$

where a_j is the jth entry of the first row of B and r_j is a solution of $r^n = 1$. More specifically for our matrix this will be

$$
\begin{aligned}
\lambda_j &= 1 + r_j + r_j^{n-1} \\
&= 1 + \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \\
&\quad + \cos\left(\frac{2(n-1)\pi}{n}\right) + i\sin\left(\frac{2(n-1)\pi}{n}\right) \\
&= 1 + 2\cos\left(\frac{2\pi}{n}\right).
\end{aligned} \tag{15}
$$

As j ranges from 0 to $n-1$ the eigenvalues will range from a maximum value of 3 to a minimum value (or values) determined by the value of $2\pi/j$ closest to π. If n is even, then for $j = n/2$ the eigenvalue will be -1; if n is odd there will be a pair of eigenvalues that are minimal. The eigenvalues given by 1 and $n-1$ will be the same as will those given by 2 and $n-2$ and so forth.

The corresponding eigenvectors are given by

$$\vec{u}_i = (1, r_i, r_i^2, \ldots, r_i^{n-1}) \tag{16}$$

from which a set of n real linear independent eigenvectors can be found, since B is real symmetric. Let us refer to this set of real eigenvectors as $\{\vec{\xi}_j\}$.

Taking the set of eigenvectors $\{\vec{\xi}_j\}$ as a basis, with $\vec{\xi}_0 = (1, 1, \ldots, 1)^T$, we can write for j between 1 and $n-1$:

$$
\begin{aligned}
c_j^{n+1}\vec{\xi}_j &= \frac{\partial f_\mu}{\partial u}(\bar{u}^n)Bc_j^n\vec{\xi}_j + O(\max_i\{(\epsilon_i^n)^2\}) \\
&= \frac{\partial f_\mu}{\partial u}(\bar{u}^n)c_j^n\lambda_j\vec{\xi}_j + O(\max_i\{(\epsilon_i^n)^2\}).
\end{aligned}
\tag{17}
$$

So if c_j^n tends to 0 for each j except for 0, the coefficient of $\vec{\xi}_0 = (1, 1, \ldots, 1)^T$, then the system will tend to the orbit of the one-dimensional dynamical system,

$$\bar{u}^{n+1} = f_\mu(\bar{u}^n) + O(\max_i\{(\epsilon_i^n)^2\}), \tag{18}$$

if the average values $\bar{u}^n$ are sufficiently close to the orbit of the one-dimensional dynamical system. This happens only if

$$\left(\prod_{i=1}^n \frac{\partial f_\mu}{\partial u}(\bar{u}^i)\right)\lambda_j^n \to 0, \tag{19}$$

as n goes to infinity. It will be sufficient for the product including λ_1 or λ_{n-1} to tend to 0 as these products will be larger than the products including the other eigenvalues.

By assumption $f_\mu(\bar{u}^n)$ has one attractive orbit, so we can see that for a value of μ the orbit of the CA will be attracted to $f_\mu(\bar{u}^n)(1, 1, \ldots, 1)^T$ if

$$\left|\frac{c_1^{n+p}}{c_1^n}\right| = \left|\lambda_1^p \prod_{j=1}^p \frac{\partial f_\mu}{\partial u}(\bar{u}^i))\right| < 1, \tag{20}$$

where p is the period of the orbit. Notice that this depends on the orbit of the one-dimensional system, in the form of the $\bar{u}^i$; and the geometry of the CA, in the form of λ_1.

This formula requires that we know the orbit of the one-dimensional system, which presents computational difficulties in practice. Let us define

$$P = \left| \prod_{i=1}^{p} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/p}, \tag{21}$$

where p is the period of the orbit of f_μ, and suppose there are two integers M_1 and M_2 with $M_2 > M_1 \gg p$ and $M_2 - M_1 \gg p$. Then we can write

$$M_2 - M_1 = kp + r, \tag{22}$$

where k is the largest integer for which $kp < M_2 - M_1$, and r is an integer with $r \ll M_2 - M_1$. We can divide to get

$$1 = \frac{kp}{M_2 - M_1} + \frac{r}{M_2 - M_1}. \tag{23}$$

If we form the product

$$\left| \prod_{i=M_1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)}$$

$$= \left| \prod_{i=M_1}^{M_1+kp} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \prod_{i=M_1+kp+1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)}$$

$$= \left| P^{kp} \prod_{i=M_1+kp+1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)}$$

$$= P^{kp/(M_2 - M_1)} \left| \prod_{i=M_1+kp+1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)}. \tag{24}$$

We see that as $M_2 - M_1$ becomes large relative to the period of the orbit p, we will have

$$P^{kp/(M_2 - M_1)} \to P \tag{25}$$

and

$$\left| \prod_{i=M_1+kp+1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)} \to 1. \tag{26}$$

So then we may approximate the inequality in equation (20) by

$$\lambda_1 \left| \prod_{i=M_1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^{i+n}) \right|^{1/(M_2 - M_1)} < 1. \tag{27}$$

Let us define a function

$$g(\mu) = \left| \prod_{i=M_1}^{M_2} \frac{\partial f_\mu}{\partial u}(\bar{u}^i) \right|^{1/(M_2-M_1)}, \tag{28}$$

and observe that this function depends only on the rule used to define the CA and not the geometry.

Using this function we may write a condition for the convergence to a state where all cells have a uniform value,

$$\lambda_1 g(\mu) < 1. \tag{29}$$

Note that the function $g(\mu)$ is determined by the rule used to assign new values and does not depend on the number of cells, so this function can be computed for the entire class of CCA that have this rule. Then the set of parameter values μ for which the CCA will have a stable orbit with the same value at all cells can be determined by satisfying equation (28).

2. Examples

In this section we demonstrate the behavior discussed in the analysis performed in section 1. Let us start with the system of CCA having new cell values defined by the rule

$$u_i^{n+1} = \mu(u_{i-1}^n + u_i^n + u_{i+1}^n)(1 - u_{i-1}^n)(1 - u_i^n)(1 - u_{i+1}^n). \tag{30}$$

The corresonding one-dimensional nonlinear dynamical system will be defined by the map

$$U^{n+1} = f(U^n) = \mu(3U^n)(1 - U^n)^3. \tag{31}$$

The map $g(\mu)$ will be defined by

$$g(\mu) = \prod_{i=M_1}^{M_2} \mu(1 - 4U^i)(1 - U^i)^2. \tag{32}$$

We need to check at this point that this map satisfies the condition of having only one attractive orbit. First let us compute the Schwarzian derivative of this function

$$\begin{aligned}
Sf &= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \\
&= -\frac{12(3 - 10U^n + 10(U^n)^2)}{(1 - 4U^n)^2(U^n - 1)^2} \\
&< 0.
\end{aligned} \tag{33}$$

Then since $Sf < 0$ and $f(U)$ has two critical points (at $U = 1/4$ and $U = 1$), we can state that f has at most four attracting periodic orbits [2].

By observing that if $U \in (-\infty, 0) \cup (1, \infty)$ then $f^k(U)$ tend to negative infinity, we can eliminate two attracting orbits leaving us the possibility that there is one orbit that attracts $1/4$ and an orbit that attracts 1. Then note that if $1 > x > f(1/4)$ then $f^n(x) < f(1/4)$ for all n, and so there is no open set W with $x \in W$ where $f(W) \subset W$, so there is only one attractive period orbit.

The finalstate diagrams for the CCA with seven cells and the one-dimensional dynamical system are illustrated in Figure 1 along with the graph of $y = 1$ and $y = \lambda_1 g(\mu)$. The finalstate diagrams were generated by taking 100 evenly spaced values of μ between 2 and 3 and iterating to approximate the orbit at each of these values. For the CCA diagram the procedure was the following. First, the one-dimensional map was iterated 1000 times to find a value close to the orbit of the one-
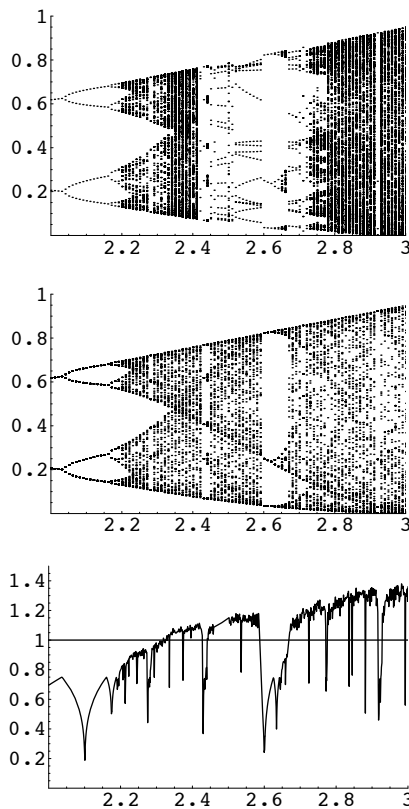

Figure 1. Finalstate diagrams for CCA with seven cells (top) and corresponding one-dimensional map (middle), $y = 1$ and $y = \lambda_1 g(\mu)$ (bottom) for parameter values of 2.0 to 3.0

dimensional system. The cells of the CCA were then initialized to this value, excepting one cell were the value was perturbed by 0.0001. The CCA were then iterated 1000 times. Following this, the cell values of the next 100 steps were plotted as a function of μ to create the diagram. This procedure is necessary to ensure that we start sufficiently close to the orbit of the one-dimensional system—while equation (28) ensures stability, it does not ensure stability for all initial states. For the one-dimensional system an initial value of 0.25 was taken, then the system was iterated for 1000 steps and the cell values of the next 100 steps were plotted to create the finalstate diagram. To generate the graph of $g(\mu)$ an initial value of 0.25 was used with $M_1 = 500$ and $M_2 = 1000$. Notice that for some values of the parameter the diagrams appear to be identical while for some values the diagrams differ. Equation (28) is satisfied for values of μ less than approximately 2.35, for a narrow band of values of μ near 2.45 corresponding to period 5 in the one-dimensional system, and for a wider band between approximately 2.6 and 2.7 corresponding to period 3. In each of these cases we can find in the CCA finalstate diagram data points that match those in the one-dimensional diagram.

Another example is illustrated by Figure 2. The same rule is used to assign the new values as in the previous example, see equation (30), except in this example there are 12 cells. The diagrams were generated in the same manner as for the case with seven cells. Again we see windows where equation (28) is satisfied, below approximately 2.25 the diagrams coincide, and between 2.5 and 3 we see the period 6 window and the coincidence of the CCA diagram and the one-dimensional system diagram. Similarly there is agreement in the period 5 window between μ values of 2.4 and 2.45 and in the period 3 window between 2.6 and 2.7.

The next example is illustrated by Figure 3. The same rule is used to assign the new values as in the previous example, see equation (30), except in this case there are 23 cells. The diagrams were generated in the same manner as for the case with seven cells. Again we see windows where equation (28) is satisfied, below approximately 2.25 the diagrams coincide, and between 2.5 and 3 we see the period 6 window and the coincidence of the CCA diagram and the one-dimensional system diagram. Similarly there is agreement in the period 5 window between μ values of 2.4 and 2.45 and in the period 3 window between 2.6 and 2.7.

Let us return to the 12-cell example and examine more closely one of the windows where the CCA diagram and the one-dimensional system diagram coincide. We will plot our diagrams for the range of parameter values from 2.42 to 2.43, taking 100 evenly spaced steps. At the top of Figure 4 we see the CCA finalstate diagram, in the middle is the one-dimensional system finalstate diagram, and at the bottom the graph of $\lambda_1 g(\mu)$. Here we can see at a finer resolution the correspondence between satisfying the condition $\lambda_1 g(\mu) < 1$ and the CCA converging to the orbit of the one-dimensional system.
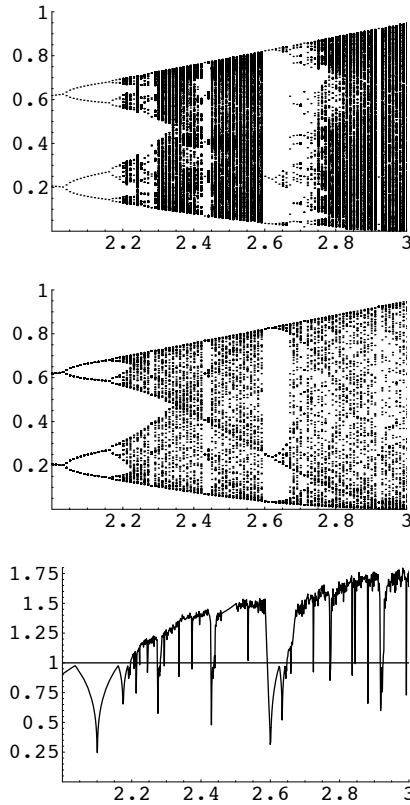
Figure 2. Finalstate diagrams for CCA with 12 cells (top) and corresponding one-dimensional map (middle), and $y = 1$ and $y = \lambda_1 g(\mu)$ (bottom) for parameter values of 2.0 to 3.0.

In our last example we will examine the effect of starting the CCA at a state that is not sufficiently close to the orbit of the one-dimensional system to ensure convergence. In Figure 5 at top we have the finalstate diagrams for the CCA initialized with a starting value of 0.23456 at all cells except one where this value was perturbed by 0.0001. In the middle we have the CCA initialized with a value near the orbit of the one-dimensional system (as in the other examples), and at the bottom there is the graph of $\lambda_1 g(\mu)$. We can clearly see the effect of the initial condition: for the CCA not near the value of the one-dimensional system the system does not converge to the orbit of the one-dimensional system for parameter values between approximately 2.4285 and 2.4295 while the CCA initialized to a value near the orbit of the one-dimensional system is stable and converges to that orbit.
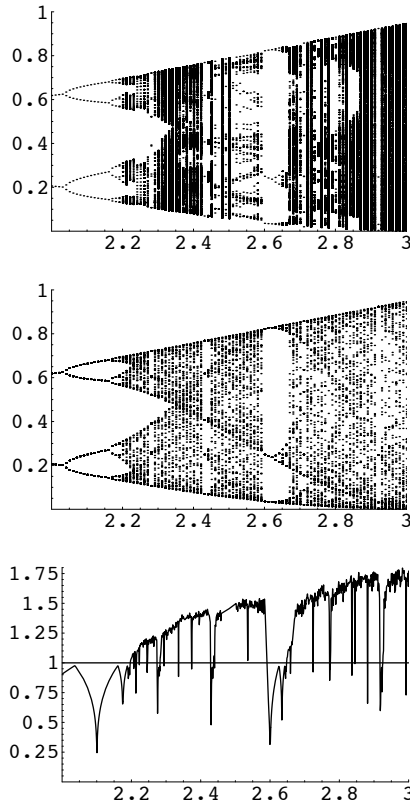
Figure 3. Finalstate diagrams for CCA with 23 cells (top) and corresponding one-dimensional map (middle), and $y = \mu$ and $y = \lambda_1 g(\mu)$ (bottom) for parameter values of 2.0 to 3.0.

3. Conclusions

Starting with a system of continuous cellular automata (CCA) we can state a stability condition for the finalstate of this system. We derive from the rule defining the CCA a one-dimensional map used to define a one-dimensional dynamical system. The rule that defines the CCA must statisfy a symmetry condition, and the one-dimensional map derived from this rule must allow only one attractive orbit. Based on the number of cells in a system of CCA a stability condition can be determined. For those parameter values that satisfy this condition the system of CCA is shown to have a stable orbit whose values are determined by the orbit of the one-dimensional system. And finally, this stable solution
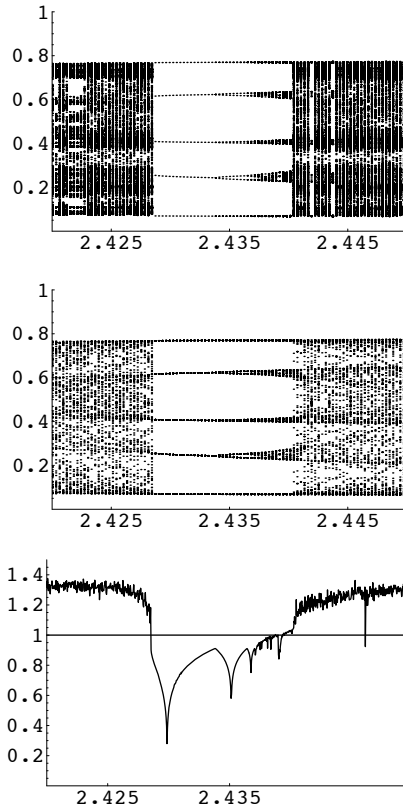
Figure 4. Finalstate diagrams for CCA with 12 cells (top) and corresponding one-dimensional map (middle), and $y = 1$ and $y = \lambda_1 g(\mu)$ (bottom) for parameter values of 2.42 to 2.43.

is guaranteed only for those initial states that are sufficiently close to the orbit of the one-dimensional system as demonstrated in one of the examples.

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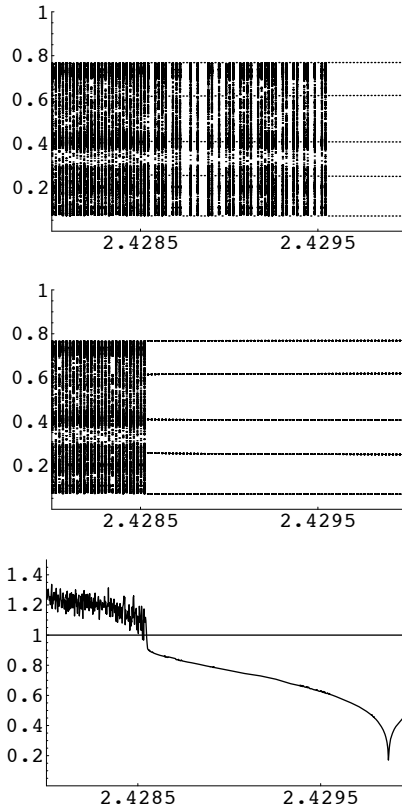[3] P. Drazin, *Nonlinear Systems* (Cambridge University Press, Cambridge, 1992).

Figure 5. CCA with 12 cells and initial state of 0.23456 (top), finalstate diagrams for CCA with 12 cells and initial state near the orbit of the one-dimensional system (middle), and $y = 1$ and $y = \lambda_1 g(\mu)$ (bottom) for parameter values of 2.428 to 2.43.

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