No Polynomial Bound for the Period of Neuronal Automata with Inhibitory Memory

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We study the sequences generated by a neuronal equation with memory of the form $x_n = \mathbf{1}[\sum_{i=1}^{k} a_i x_{n-i} - \theta]$, where k is the size of the memory. We show that in the case where all the parameters $(a_i)_{1 \le i \le k}$ are negative real numbers, there exists a neuronal equation of memory length k that generates a sequence of period $e^{\Omega(\sqrt{k \log k})}$. This result shows that in the case where all weighting coefficients are negative, the neuronal recurrence equation exhibits a complex behavior.

1. Introduction

In [1] it is suggested that the dynamic behavior of a single neuron with memory that does not interact with other neurons can be modeled by the following recurrence equation:

$$x_n = \mathbf{1}\left[\sum_{i=1}^{k} a_i x_{n-i} - \theta\right] \tag{1}$$

where we have the following.

- x_n is a boolean variable representing the state of the neuron at time $t = n$.
- k is the memory length, that is to say, the state of the neuron at time $t = n$ depends on the states assumed by the neuron at the k previous steps $t = n - 1, \ldots, n - k$.
- The values a_i $(i = 1, \ldots, k)$ are real numbers called the *weighting coefficients*; a_i represents the influence of the state of the neuron at time $n - i$ on

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the state of the neuron at time n. That influence is said to be *excitatory* if $a_i > 0$, *inhibitory* if $a_i < 0$, and *null* if a_i is equal to zero.

- θ is a real number called the *threshold*.
- $\mathbf{1}[u] = 0$ if $u < 0$, and $\mathbf{1}[u] = 1$ if $u \geq 0$.

The system obtained by interconnecting several neurons is called a neural network (NN). Such networks were introduced in [4], and are quite powerful. Indeed, it can be shown that they can be used to simulate any Turing machine. More recently, NN have been studied extensively as tools for solving various problems such as classification, speech recognition, and image processing [5]. The application field of the threshold functions is large. The spin moment of the spin glass is one of the widest examples in solid state physics that have been simulated by NN. In electricity, for instance, a threshold function represents a transistor; in social science a threshold function is often used to represent vote laws.

Let p and T be two positive integers such that $p > 0$ and $T \geq 0$. Equation (1) is said to be of period p and transient T if and only if:

- $\forall\, t, t' \in \mathbb{N};\ t, t' \in \{k-1, \ldots, p+T-1\}\ \ t \neq t'$ implies that $Y(t) \neq Y(t')$
- $Y(p+T) = Y(T)$

where $Y(t) = (x_t, x_{t-1}, \ldots, x_{t-k+2}, x_{t-k+1})$. The period and transient of sequences generated by a neuron are good measures of the complexity of the behavior of the neuron.

Let us denote $LP(k)$ as the longest period that can be generated by a neuronal equation with memory length k. In [2], it was conjectured that if $(a_i)_{1\leq i\leq k} \in \mathbb{R}$, then $LP(k) \leq 2k$. This conjecture has been disproved. The best lower bound in $LP(k)$ is $O(e^{\sqrt{k}\log k})$, which was proved in [7]. In [2], it was also conjectured that if $\forall\, i, i = 1, \ldots, k$, $a_i \in \mathbb{R}^+$ (i.e., $a_i \geq 0$), then $LP(k) \leq k$. This conjecture has been disproved in [8] where a neuronal recurrence equation of memory length k and of period $O(k^3)$ has been exhibited.

When all the weighting coefficients are negative, the influence of the previous states of a neuron (at time $n-k, n-k+1, \ldots, n-2, n-1$) on its state at time n is inhibitory, and from a physiological point of view, it is important to know the behavior of that class of neurons. In [6], the behavior of recurrence neuronal equations of length 3, 4, 5, and 6 are studied. In the case where the memory length is 5 or 6 and all the weighting coefficients are negative, many cycles of length less than or equal to 29 are exhibited.

In this paper, we exhibit a neuronal recurrence equation of memory length h where all the weigthing coefficients are strictly negative that generates a sequence of period $e^{\Omega(\sqrt{h\log h})}$.

2. Neuronal recurrence equation with negative weighting coefficients

Let k be a positive integer. For a vector $a \in \mathbb{R}^k$, a real number $\theta \in \mathbb{R}$, and a boolean vector $w \in \{0,1\}^k$ we define the boolean sequence $(x_n)_{n\in\mathbb{N}}$ by the following recurrence

$$x_t = \begin{cases} w_t & t \in \{0,\dots,k-1\} \\ \mathbf{1}([a,x]_t - \theta) & t \geq k \end{cases} \tag{2}$$

where $[a,x]_t = \sum_{i=1}^{k} a_i x_{t-i}$.

We denote by $S(a,\theta,w)$ the sequence generated by equation (2) and by $T(a,\theta,w)$ its period.

Let m be a positive integer and $\bar{\theta}$ a real number satisfying $\bar{\theta} \geq 2m$. We define the following sets:

$$\begin{aligned} E &= \{3m - i : i = 1,\dots,m\} = \{2m, 2m+1, \dots, 3m-1\} \\ H &= \{6m - 2i : i = 1,\dots,m\} = \{4m, 4m+2, \dots, 6m-2\} \\ F &= E \cup H \\ G &= \{1,2,\dots,6m\} \setminus F. \end{aligned}$$

Let $\bar{a} \in \mathbb{R}^{6m}$ be the vector defined by

$$\bar{a}_i = \begin{cases} \bar{\theta}/2 - 3m + i & \text{if } i \in E \\ \bar{\theta}/2 + 3m - i/2 & \text{if } i \in H \\ -6m(\bar{\theta} + m) & \text{otherwise.} \end{cases} \tag{3}$$

For every j in $\{1,\dots,m-1\}$ let w^j be the boolean vector defined by $w^j_i = 0$ for $i \in \{0,\dots,6m-1\} \setminus \{2j, 3m+j\}$ and $w^j_{2j} = w^j_{3m+j} = 1$.

Property 1 of Lemma 1 was proved in [10] and Property 2 of Lemma 1 can be easily deduced from the evolution of the sequence $S(\bar{a},\bar{\theta},w^j)$.

Lemma 1. The sequence $x^j = S(\bar{a},\bar{\theta},w^j)$ satisfies the following properties.

1. $T(\bar{a},\bar{\theta},w^j) = 3m - j$.
2. For every $j = 1,\dots,m-1$ and for every $t \geq 6m$ we have:
   (a) $2 \leq \sum_{i\in F} x^j_{t-i} + \sum_{i\in G} x^j_{t-i} \leq 3$
   (b) $\sum_{i\in F} x^j_{t-i} + 2\sum_{i\in G} x^j_{t-i} \geq 2$
   (c) If $x^j_t = 1$ then $\sum_{i\in F} x^j_{t-i} = 2$ and $\sum_{i\in G} x^j_{t-i} = 0$.

By modifying the coefficients $\bar{a}_i$ we have the following result.

Theorem 1. There exist $b \in \mathbb{R}^{6m}$ with $b_i < 0$ for every $i = 1,\dots,6m$, and $\theta \in \mathbb{R}$ such that for every $j = 1,\dots,m-1$ we have that $S(b,\theta,w^j) = S(\bar{a},\bar{\theta},w^j)$.

Proof. Let λ be a negative real number. For each $i = 1, \ldots, 6m$ we define b_i by the following equation

$$b_i = \begin{cases} \bar{a}_i + \lambda & \text{if } i \in F \\ \bar{a}_i + 2\lambda & \text{if } i \in G \end{cases} \tag{4}$$

and

$$\theta = \bar{\theta} + 2\lambda.$$

In order to prove that the sequences $(y^j) = S(b, \theta, w^j)$ and $(x^j) = S(\bar{a}, \bar{\theta}, w^j)$ are the same we apply induction on t. Let us assume that $y_i^j = x_i^j$ for all $i < t$.

Using the definition of (b_i) in terms of $(\bar{a}_i)$ we have that

$$\begin{aligned} [b, y^j]_t - \theta &= \sum_{i=1}^{k} b_i y_{t-i}^j - \theta \\ &= [\bar{a}, x^j]_t - \bar{\theta} + \lambda \left(\sum_{i \in F} x_{t-i}^j + 2 \sum_{i \in G} x_{t-i}^j - 2 \right). \end{aligned} \tag{5}$$

From Lemma 1 we have that

$$\sum_{i \in F} x_{t-i}^j + 2 \sum_{i \in G} x_{t-i}^j - 2 \geq 0.$$

Since $\lambda < 0$ we conclude that $[b, y^j]_t - \theta \leq [\bar{a}, x^j]_t - \bar{\theta}$. From this inequality we deduce that if $x_t^j = 0$ then $y_t^j = 0$ and from Lemma 1 we know that if $x_t^j = 1$ then $\sum_{i \in F} x_{t-i}^j = 2$ and $\sum_{i \in G} x_{t-i}^j = 0$. In this situation $[b, y^j]_t - \theta = [\bar{a}, x^j]_t - \bar{\theta}$ and then the conclusion follows. ■

From [3] we have the following fundamental lemma of composition of a neuronal recurrence equation with memory.

Lemma 2. For $b \in \mathbb{R}^k$ and $\theta \in \mathbb{R}$ let $T_1, \ldots, T_s$ be the different periods $T(b, \theta, w^j)$ when w^j varies over $\{0,1\}^k$. Then, there exist $c \in \mathbb{R}^{ks}$ and $v \in \{0,1\}^{ks}$ such that $T(c, \theta, v) = s \cdot \operatorname{lcm}\{T_1, \ldots, T_s\}$.

Proof. For every $i = 1, \ldots, k$ we define $c_{is} = b_i$ and $c_j = 0$ if $j \neq is$. For every $i = 0, \ldots, k-1$ and every $j = 0, \ldots, s-1$ we define $v_{is+j} = w_i^j$ where $T(b, \theta, w^j) = T_j$. ■

Let m be a positive integer. We denote by $\rho(m)$ the cardinality of the set $\mathcal{P} = \{p : p \text{ prime}, 2m < p < 3m\}$. Let us denote $p_1, \ldots, p_{\rho(m)}$ as the prime numbers within $\{2m, \ldots, 3m\}$ and $\lambda(m) = \prod_{i=1}^{\rho(m)} p_i$. Finally, we define $h(m) = 6m\rho(m)$.

Corollary 1. For every positive integer m there exist $c \in \mathbb{R}^{h(m)}$, $\theta \in \mathbb{R}$, and $v \in \{0,1\}^{h(m)}$ such that $c_i \leq 0$ for every $i = 1, \ldots, h(m)$ and with $T(c, \theta, v) = \rho(m)\lambda(m)$.

Proof. From Lemma 1 and Theorem 1 we know that for $j \in \mathcal{P}$ we have that $T(b, \theta, w^{3m-j}) = j$ with $b_i < 0$ for every $i = 1, \ldots, 6m$. We construct the vector c as in Lemma 2. By construction the vector c satisfies $c_i \leq 0$, for $i = 1, \ldots, h(m)$. From w^{3m-j} with $j \in \mathcal{P}$ we construct v as in Lemma 2. Then $T(c, \theta, v) = \rho(m)\lambda(m)$. ■

The technique used in Corollary 1 defines several coefficients c_i as zero. We will show that it is possible to modify the coefficients (c_i) so as to obtain the previous result with all coefficients being negative.

Lemma 3. For every $c \in \mathbb{R}^k$ and $\theta \in \mathbb{R}$ with $c_i \leq 0$, for $i = 1, \ldots, k$ there exist $d \in \mathbb{R}^k$ and $\theta' \in \mathbb{R}$ such that for every $i = 1, \ldots, k$, $d_i < 0$ and for every $w \in \{0,1\}^k$ we have that $S(c, \theta, w) = S(d, \theta', w)$.

Proof. It suffices to prove that there exist $d \in \mathbb{R}^k$, $\theta' \in \mathbb{R}$ such that for every $i = 1, \ldots, k$, $d_i < 0$ and for every $y \in \mathbb{R}^k$, we have:

$$\mathbf{1}\left[\sum_{i=1}^{k} d_i y_i - \theta'\right] = \mathbf{1}\left[\sum_{i=1}^{k} c_i y_i - \theta\right].$$

Let μ_1 and μ_2 be defined by

$$\mu_1 = \max\left\{\sum_{i=1}^{k} c_i y_i : \sum_{i=1}^{k} c_i y_i < \theta,\ y \in \{0,1\}^k\right\}$$

and

$$\mu_2 = \min\left\{\sum_{i=1}^{k} c_i y_i : \sum_{i=1}^{k} c_i y_i \geq \theta,\ y \in \{0,1\}^k\right\}.$$

We define

$$\theta' = \theta - \frac{\theta - \mu_1}{2} = \frac{\theta + \mu_1}{2}.$$

Then $\mu_1 - \theta' < 0 < \mu_2 - \theta'$. We define

$$d_i = c_i - \frac{\mu_2 - \theta'}{k}.$$

Clearly every coefficient d_i is negative and we have that

$$\sum_{i=1}^{k} d_i y_i - \theta' = \sum_{i=1}^{k} c_i y_i - \left(\frac{\mu_2 - \theta'}{k}\right)\sum_{i=1}^{k} y_i - \theta'.$$

Since $0 \leq \sum_{i=1}^{k} y_i \leq k$ we get that

$$\sum_{i=1}^{k} c_i y_i - \mu_2 \leq \sum_{i=1}^{k} d_i y_i - \theta' \leq \sum_{i=1}^{k} c_i y_i - \theta'.$$

Therefore, if $\sum_{i=1}^{k} c_i y_i < \theta'$ we obtain $\sum_{i=1}^{k} d_i y_i < \theta'$. Since $\theta' \geq \mu_1$, from the definition of μ_1 and μ_2 we know that if $\sum_{i=1}^{k} c_i y_i \geq \theta'$ then $\sum_{i=1}^{k} c_i y_i \geq \mu_2$, which implies that $\sum_{i=1}^{k} d_i y_i \geq \theta'$. ■

Corollary 2. For every m there exist $d \in \mathbb{R}^{h(m)}$, $\theta' \in \mathbb{R}$, and $v \in \{0,1\}^{h(m)}$ such that $T(d, \theta', v) = e^{\Omega\left(\sqrt{h(m)\log h(m)}\right)}$.

Proof. From Lemma 3 we know that there exist $d \in \mathbb{R}^{h(m)}$, $\theta' \in \mathbb{R}$, and $v \in \{0,1\}^{h(m)}$ such that $T(d, \theta', v) = \rho(m)\lambda(m)$. We prove that $\lambda(m) = e^{\Omega\left(\sqrt{h(m)\log h(m)}\right)}$. It is known that

$$\lim \frac{\pi(m)}{\frac{m}{\log(m)}} = 1,$$

where $\pi(m)$ is the number of prime numbers less than m. Then, for m large enough we have that

$$\rho(m) \sim \frac{3m}{\log(3m)} - \frac{2m}{\log(2m)} \sim \frac{m\log(m)}{\log^2(m)} = \frac{m}{\log(m)}.$$

From that we deduce that $h(m) \sim m^2/\log(m)$ and then

$$m \sim \sqrt{h(m)\log h(m)}.$$

Using the bounds given in [9] we have for x large enough that

$$e^{0.9x} \leq \gamma(x) \leq e^{1.1x}$$

where $\gamma(x) = \prod_{p \leq x,\ p \text{ prime}} p$. Therefore,

$$e^{0.5m} = e^{2.7m - 2.2m} \leq \lambda(m) \leq e^{3.3m - 1.8m} = e^{1.5m}$$

we conclude that

$$\lambda(m) = e^{\Omega\left(\sqrt{h(m)\log h(m)}\right)}. ■$$

3. Conclusion

The technique used to pass from a neuronal recurrence equation where the weighting coefficients are positive or negative to the neuronal recurrence equation where all the weighting coefficients are negative is inscribed in the frame of structural constructions. The existence of a neuronal recurrence equation of memory length k which describes a cycle of length $e^{\Omega\left(\sqrt{k \log k}\right)}$ shows that the behavior of neuronal recurrence equations is complex when all the weighting coefficients are negative.

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