Complex Systems **10** (1996) 219–227

Complexity Growth in Almost Periodic Fluids in the Case of Lattice Gas Cellular Automata and Vlasov Systems

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Abstract. Two examples of dynamical systems with spatial almost periodicity are considered and the complexity growth during the time evolution is quantified.

The first example deals with almost periodic cellular automata (CA). The growth rate of the information which is needed to store a configuration can quantitatively be given by a real number which is bounded above by the dimension of the CA.

In the second example, where periodicity is understood in the classic sense of Bohr, the collision-free Boltzmann equation is considered. The method of characteristics allows the proof of an existence theorem for almost periodic initial conditions. A Lyapunov exponent measures the growth of complexity during the evolution.

1. Introduction

Generally speaking, any physical system that can have periodic boundary conditions can also be considered in an almost periodic setup. Almost periodicity allows averaging and yields global translational invariant quantities. While this is also possible in more general ergodic setups, almost periodicity has the feature of strict ergodicity so that the mean is defined by a single configuration alone. The hull, the closure of all translates of an almost periodic configuration, is a compact topological group on which averaging with respect to the unique Haar measure gives macroscopic quantities. Almost periodic configurations can have more than the obvious translation symmetry because the dimension of the hull of the function is in general larger than the space on which the functions are defined. The other modes of symmetry are internal symmetries of the system.

Almost periodicity appears in the literature mostly for systems with an almost periodic time-dependent forcing term or when almost periodic solutions of the systems are considered. This is not the topic here. We consider particle configurations which have spatial almost periodicity and evolve them with a time-independent law. Actually, we are interested in situations where the complexity grows during the evolution, making it impossible that the solution is also almost periodic in time.

In section 2, we reconsider almost periodic lattice gas automata, for which almost periodicity is understood in the sense of minimality. In section 3, we look at spatial Bohr almost periodic Vlasov systems in which context spatial almost periodicity appears to be new. In both situations, the time evolution renders the configuration increasingly hard to describe and there is a measure for the growth of complexity.

First, we mention three other spatially almost periodic dynamical systems, which belong to the same type we are interested in but which we do not discuss any further in this paper.

1. Almost periodic KdV and Toda systems. The almost periodic Korteweg-de Vries (KdV) equation introduced in [12] is an example where spatial almost periodicity appears in fluid dynamics. It is defined as the isospectral deformation of a one-dimensional almost periodic Schrödinger operator $L = -\Delta + V$, where V is a Bohr almost periodic function. It is a model for an almost periodic fluid in a shallow water channel. If V is periodic, one obtains the KdV equation with periodic boundary conditions as a special case. A discrete version of the almost periodic KdV is the almost periodic Toda system [7]

$$\ddot{f} = \exp(f(x+\alpha) - f(x)) - \exp(f(x) - f(x-\alpha))$$

on the space of continuous periodic functions f. It describes a chain of oscillators with position $f_n = f(x+n\alpha)$ which are nearest neighbor coupled by an exponential potential:

$$\ddot{f}_n = \exp(f_{n+1} - f_n) - \exp(f_n - f_{n-1}).$$

If $\alpha = p/q$ is rational, then $f_{n+q} = f_n$ and the system is the periodic Toda lattice. It can be integrated by conjugating the flow to a linear flow on the Jacobi variety of a Riemann surface defined by the isospectrally deformed Jacobi matrix. In the almost periodic case, one still has an isospectral deformation of an almost periodic Jacobi matrix. However, an infinite dimensional generalization of the algebro-geometric integration is expected to work only in very special cases.

2. Almost periodic discrete parabolic or hyperbolic PDEs. Coupled map lattices are dynamical systems which are used to model partial differential equations (PDEs). They can be viewed as "CA with a continuum alphabet." Such systems appear in numerical codes for PDEs. They became popular especially with the work of Kaneko, Bunimovich, and Sinai because they provide systems with "space-time chaos" which has an easy proof using an idea of Aubry [6]. An almost periodic example of a coupled map lattice is the evolution $f \mapsto \phi(f)(x) = \epsilon \sum_{i=1}^{d} (f(x+\alpha_i)+f(x-\alpha_i))+V(f(x))$ on the space of continuous functions on the d-dimensional torus, where $x \mapsto x + \alpha_i$ are translations. The aperiodic configuration $f_n = f(n \cdot \alpha)$ with $n \in \mathbf{Z}^d = \{n = \sum_{i=1}^{d} n_i e_i \}$ evolves in time according to $f_n \mapsto \epsilon \sum_{j=1}^{d} (f_{n+e_j} + f_{n-e_j}) + V(f_n)$. The name "coupled map lattice" comes from the fact that for $\epsilon = 0$, the system is an array of decoupled maps $f_n \mapsto V(f_n)$, which become coupled when $\epsilon > 0$. Again, if all α_i are rational, these configurations are periodic, if α is irrational the configurations are almost periodic. Not only parabolic PDEs but also hyperbolic PDEs such as nonlinear wave equations, have discrete analogues as symplectic coupled map lattices. An example is $(f_t, g_t) \mapsto (f_{t+1}, g_{t+1}) = (\epsilon[\sum_{i=1}^{d} f(x+\alpha_i)+f(x-\alpha_i)]+V(f(x)), f_t)$ on pairs of periodic functions on the torus. This discrete PDE can be rewritten as $f_{t+1} - 2f_t + f_{t-1} = \epsilon \sum_{i=1}^{d} (f_t(x+\alpha_i) - f(x) + f_t(x-\alpha_i)) + W(f_t(x))$ with $W(f) = (V(f(x)) - 2f + 2d\epsilon f)$. This is a discrete version of a nonlinear wave equation because $f_{t,n} = f_t(n \cdot \alpha)$ satisfies the discrete nonlinear wave equation $f_{t+1,n} - 2f_{t,n} + f_{t-1,n} = \epsilon \sum_{i=1}^{d} (f_{t,n+e_i} - 2f_{t,n} + f_{t,n-e_i}) + W(f_{t,n})$, a discretization $f_{tt} = \epsilon \Delta f + W(f)$. For $\epsilon = 0$, it is an array of decoupled Henon type twist maps. For a cubic polynomial W, it occurs as the Euler equations of a natural functional [8].

3. Almost periodic Riemannian geometry and Vlasov–Einstein dynamics. Almost periodicity is also interesting in a differential geometric setup, where interacting particles move along geodesics. An almost periodic metric g on Euclidean space defines an almost periodic Riemannian manifold on which one can average. In some sense, such a manifold looks like a torus because the mean of the curvature gives zero. Averaging through almost periodicity could be interesting in general relativity, because the Hilbert action is still defined by an almost periodic mean. Without a compactness assumption on the manifold, this would only be possible by assuming asymptotic flatness of the metric. A metric solving the almost periodic Einstein equations is a critical point of a well defined variational problem. The classic Vlasov equation considered here has as a relativistic analogue the Vlasov–Einstein equation $y^j \nabla_{x_j} P - \Gamma_{ij}^k y^i y^j \nabla_{y_k} P = 0$, which describes matter not interacting through a potential but through the metric: the connection Γ is determined from a metric g solving the Einstein equations $G(g) = 8\pi T(P)$. Existence results are only known in very special situations. Solving the Einstein equations in the almost periodic case and a better understanding of the geodesic flow in an almost periodic metric are problems that have not yet been addressed.

2. Almost periodic lattice gas automata

Lattice gas CA are used for numerical simulations of fluids. They provide a reasonable numerical method when the fluid has complex geometric boundary conditions. Almost periodic CA [4] can store and process large periodic configurations in a compressed form and allow for dealing with infinite aperiodic configurations. The size of the data used to store the configuration measures the complexity of the fluid. The growth rate of this information is a numerical quantity which can be determined. As in the case of the entropy for smooth dynamical systems, one cannot expect to compute it explicitly in general.

The term "almost periodicity" for CA is used as a synonym for minimality, a common terminology in mathematical physics that is different from the mathematical term of Bohr almost periodic functions considered in section 3: a configuration $x \in A^{\mathbf{Z}^d}$ is called almost periodic if no nontrivial closed shift invariant subset in the closure of the orbit of x exists. Almost periodicity is interesting since it occurs in nature, for example, in quasicrystalline materials, where the centers of the atoms form almost periodic configurations. Mathematically, almost periodic initial conditions are a natural alternative to periodic or random initial conditions.

CA are dynamical systems which evolve a d-dimensional array of letters $a \in A$ in a finite alphabet $A = \{0, 1, \dots, N-1\}$ using a translational invariant rule. More precisely, a CA is a continuous map ϕ on the compact space $X = A^{\mathbf{Z}^d}$ which commutes with all translations $x \mapsto (\sigma_k(x))_n = x_{n+k}$ with $k \in \mathbf{Z}^d$. By the Curtis–Hedlund–Lyndon theorem, the new state $\phi(x)_n$ at a grid point $n \in \mathbf{Z}^d$ depends only on $\{x_k\}_{k \in n+F}$, where F is a finite neighborhood of n. As for any dynamical system ϕ, one is interested in invariant sets such as attractors or periodic orbits. Such invariant sets are crucial for understanding the long term behavior of the dynamics and especially the behavior of averaged quantities over time. Since $\phi(X) \subset X$, also $\phi^{n+1}(X) \subset \phi^n(X)$ and the attractor $K = \bigcap_{n \in \mathbf{N}} \phi^n(X)$ is a compact set on which all invariant measures of ϕ have their support. K decomposes in general to smaller closed invariant sets. The minimal invariant subsets are by definition almost periodic and sometimes, we expect a minimal set that can be given by a Sturmean configuration which can be stored with a small amount of data. At a fixed time such a configuration is represented as a union of half open intervals $J = \bigcup_{i=1}^m [a_i, b_i)$. Associated with each interval $J_i = [a_i, b_i)$ is a value $f(J_i) \in A$. For $\theta \in \mathbf{R}^n$, a Sturmean configuration is given by

$$x_n = \sum_i f(J_i) 1_{J_i}(\theta + n \cdot \alpha \bmod 1),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a vector of d rotation numbers, θ is a parameter, and $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$ is the location of a cell in the lattice $\mathbf{Z}^d$. Here $\theta \mapsto 1_H(\theta)$ is the characteristic function of a set H, which is 1 for $\theta \in H$ and 0 for $\theta \notin H$. Periodicity can be established by taking rational numbers α_i. It can be seen in a constructive way that the CA rule ϕ applies to the interval data J. The translation back to the lattice $J \mapsto x$ is only necessary when inspecting part of the phase space. Note also that with a nondeterministic rule there is still an evolution on the set of intervals.

An interesting quantity is the growth rate of the number $|\phi^n(J)|$ of intervals. It is a measure for the memory needed to store a configuration.

Of course, if the numbers α_i are all rational, then there is an upper bound for the number of intervals. If one rotation number α_i is irrational, the number of intervals will in general grow indefinitely. How fast can it grow? Let $F(\phi) \subset \mathbf{Z}^d$ be the influence region of ϕ and assume $[0,1) = \bigcup_{k=0}^{N-1} J_k$, where each J_k is a finite union of half open intervals $J_k = \sum_{j=1}^{n_k} [a_{jk}, b_{jk})$ with $a_{jk} < b_{jk}$. An interval configuration J consists of $|J| := \sum_{k=0}^{N-1} n_k$ half open intervals. We claim that the number of intervals $|\phi^n(J)|$ has a polynomial upper bound $|\phi^n(J)| \leq |F(\phi^n)| \cdot |J|$. To prove this it is enough to show the claim for $n = 1$ and then replace ϕ by ϕ^n. If $B = \{a_1, \dots, a_{|J|}\}$ with $0 \leq a_1 < a_2 \cdots < a_{|J|} < 1$ is the set of boundary points of the partition J, then the set of boundary points of the new partition $\phi(J)$ is contained in the set $B + F \cdot \alpha$ which contains not more than $|F(\phi)| \cdot |B| = |F(\phi)| \cdot |J|$ elements. If ϕ has radius r, then ϕ^n has a radius $\leq nr$ and $|F| \leq (2r+1)^d$. We can therefore define a growth rate of complexity

$$\lambda(J) := \limsup_{n \to \infty} \frac{\log |\phi^n(J)|}{\log(n)} .$$

It satisfies $\lambda(J) \leq d$ because $|F(\phi^n)| \leq (2nr+1)^d$. The actual growth rate can of course be smaller. See [4] for examples where $J = \phi^n(J)$ and $\lambda(J) = 0$. From all of the numerical experiments that we have performed, it is reasonable to believe that the limit actually exists. Of course, if α_i are all rational then $\lambda(J) = 0$. In this case, the complexity bound shows that evolving intervals is as efficient as evolving CA traditionally. It is in general better, because evolving the intervals performs the CA computations in a compressed way. From the information point of view we should look at the number of intervals as a measure of complexity because it is an upper bound for the Chaitin complexity of the configuration. If $\lambda(J) > 0$ the dynamics are not recurrent even though they are reversible.

The number λ seems not to be related to the entropy defined in [9] because looking at the CA dynamics on a thin subset of almost periodic configurations changes the dynamical properties considerably. For example, the one-dimensional shift on $\{0,1|\}^{\mathbf{Z}}$ has entropy $\log(2)$, while in the almost periodic setup $\lambda(J) = 0$ for every J because $\phi(J) = (J + \alpha) \bmod 1$ implies that the number of intervals does not change. Let us look at the deterministic and reversible lattice gas automaton HPP [2, 3], where up to four particles can be at a cell moving along one of the basis vectors $\pm e_1, \pm e_2$. During an iteration step, a particle is advanced one cell in the direction of its velocity. When two particles collide with opposite velocity, both particles are scattered by a rotation of 90 degrees. This rule preserves the particle number, the total momentum, and the total energy, quantities that are also defined for aperiodic almost periodic situations. Note that the automaton is reversible because ϕ has an inverse which is obtained by evolving the particle configuration, where the directions of the particles are reversed. The theoretical bound gives $|\phi^n(J)| \leq |J| \cdot (2n+1)^2$ and $\lambda \leq 2$. Experiments with almost periodic lattice gas automata were done in [4, 11]. We measured values $\lambda(J) = 1.95$ for the HPP model. The value $\lambda(J)$ seems to be quite independent of the initial condition J. If $\lambda > 0$, the dynamics are not recurrent even though the system is reversible. We observe a weak convergence of the measures $\sum_{f(\phi^n(J)_i)=a} 1_{\phi^n(J)_i} dx$ to a multiple of the Lebesgue measure. This uniform mixing of intervals is expected to hold whenever $\lambda(J) > 0$ and indicate that the system does approach equilibrium, a fact which is expected to happen for some CA used in fluid dynamics [15].

3. Almost periodic particle dynamics in the Vlasov limit

In this section almost periodicity appears in a different way, particles no longer move on a discrete grid as before but as a gas in $\mathbf{R}^d$ in the Vlasov limit (see [5, 13]). We prove here an existence result in the almost periodic context. Vlasov dynamics are used especially in stellar dynamics and plasma physics.

A finite dimensional system of particles moving on a manifold $N = \mathbf{R}^d$ under a pairwise interaction given by a potential V evolve according to the Newton equations $\dot{f}_j = g_j$, $\dot{g}_j = -n^{-1} \sum_{i=1}^n \nabla V(f_j - f_i)$ which are the Hamilton equations with Hamiltonian $H(f,g) = \sum_{j=1}^n g_j^2/2 + \sum_{i \neq j} V(f_i - f_j)$. We assume that the potential V is smooth and that the solutions exist for all times. If the force is rescaled in the limit $n \to \infty$ so that it stays finite, then the dynamics can be extended from point particles to a "particle gas" with an arbitrary density m in the phase space $S = \mathbf{R}^{2d}$. One evolves then a map $X = (f,g) : S \to S$, where (f_t, g_t) gives the position and momentum of the particle with initial condition (f_0, g_0). The corresponding mean-field characteristic equations

$$\dot{f} = g,\ \dot{g} = -\int \nabla V(f(\omega) - f(\eta))\ dm(\eta)$$

are the Hamilton equation for the Hamiltonian

$$H(f,g) = \int_S \frac{g(\omega)^2}{2}\ dm(\omega) + \int_{S \times S} V(f(\omega) - f(\eta))\ dm(\omega) dm(\eta).$$

This is an ordinary differential equation (ODE) in an affine linear space of all continuous functions $(f,g)(\omega) = \omega + (F,G)$ with supremum norm for $F, G \in C(S,S)$. By the Cauchy–Piccard existence theorem for ODEs in Banach spaces and a Gronwall estimate, there is a unique global solution if the gradient ∇V is smooth and bounded. The density $P_t = (f_t, g_t)_* m$ defined by $\int h(x,y)\ dP_t(x,y) = \int h(f(\omega), g(\omega))\ dm(\omega)$ satisfies then the Vlasov equation

$$\dot{P}_t(x,y) + y \cdot \nabla_x P_t(x,y) - \left[\int_S \nabla_x V(x - x') P_t(x', y')\ dx' dy' \right] \cdot \nabla_y P_t(x,y) = 0.$$

This method of characteristics [1] is a convenient way to prove the existence and uniqueness of the solution of this integro PDE.

We now extend this setup. How general can the initial density $m = P_0$ be? It can be a finite measure or a signed measure representing charged particles with different charges. One restriction is that $-\int_M \nabla V(f(\omega) - f(\eta))\, dm(\eta)$ should be finite. Only if ∇V decays sufficiently fast at infinity can one allow spatial infinite measures like the product of the Lebesgue measure on $N = \mathbf{R}^d$ with a compactly supported measure. For periodic V and when the position space N is the torus $\mathbf{T}^d$, one deals with particles in a box with periodic boundary conditions. Integration over $\mathbf{T}^d$ gives a finite force. We generalize this now to the almost periodic case, where we evolve a gas on $\mathbf{R}^d$ for which all physical quantities are almost periodic in the position.

We assume that the initial density $m(x, y) = P_0(x, y)$ on the phase space $S = T^*N$ has the property that for any continuous function h on $\mathbf{R}^d$, the function $L^h(x) = \int_{\mathbf{R}^d} h(y)m(x, y)dy$ is a Bohr almost periodic function on N and that there exists a constant r such that $m(x, y) = 0$ for $|y| > r$. To define a Vlasov evolution for such measures, we proceed in a similar way as before. Define $(f, g)(x, y) = (x+F(x, y), y+G(x, y))$, where $F, G : S \to S$ are continuous with the property that they are Bohr almost periodic functions in x when y is fixed. Such functions form a closed subspace of $C(S, S)$ on which a finite mean

$$M[F] = \lim_{n\to\infty} (2n)^{-d} \int_{[-n,n]^d\times\mathbf{R}^d} F(x, y)\, m(x, y)dxdy$$

is defined. Especially, when m is 1-periodic in x, one has

$$M[F] = \int_{\mathbf{T}^n\times\mathbf{R}^d} F(x, y)m(x, y)\, dxdy.$$

An evolution can now be defined with the Hamilton equations

$$\dot{F}(x, y) = G(x, y),\ \dot{G}(x, y) = -M[\nabla V(f(x, y) - f(\cdot))] = Z(F)(x, y)$$

with initial conditions $(F_0, G_0) = (0, 0)$. The Hamiltonian is $H(F, G) = M[G(\omega)^2/2] + M[M[V(f(\omega) - f(\eta))]]$. Because $\nabla_{x_i} V(f(x, y) - f(x', y'))$ is both almost periodic in x and x', we know that $(\dot{F}, \dot{G}) = (G(x, y), Z(F)(x, y))$ is almost periodic in x. The map $F \mapsto Z(F)$ is differentiable so that by the Cauchy–Piccard existence theorem, there is a solution in the Banach algebra of almost periodic functions for small times. A Gronwall estimate assures global existence of the solution if the gradient ∇V satisfies a global Lipshitz estimate. The corresponding Vlasov equation defines the evolution of spatial almost periodic measures $P_t = (f_t, g_t)^* P_0$, where $P_0 = m$ is the initial measure. For any $h : \mathbf{R}^d \to \mathbf{R}$, the function $L^h_t(x) = \int_{\mathbf{R}^d} h(y)P_t(x, y)\, dy$ on N is almost periodic. Examples are the physically relevant moments $L^h_t(x) = \int_{\mathbf{R}^d} \prod_i y_i^{n_i} P_t(x, y)\, dy$.

The Fourier transform of the almost periodic function $L^h_t(x)$ is a discrete measure $\hat{L}^h_t(\lambda)$ on $N = \mathbf{R}^d$. One writes

$$L^h_t(x) = \sum_{\lambda, M[L^h_t \exp(-i\lambda\cdot x)]\neq 0} \hat{L}^h_t(\lambda) \exp(i\lambda \cdot x).$$

The measure $\hat{L}^h_t$ is supported on the frequency module of the initial measure $m = P_0$. This generalizes the fact that if $P_0(x, y)$ is periodic in x, then

$P_t(x,y)$ is also periodic with the same period. While the frequency module does not change under the evolution, the weights on the spectrum change and are expected to shift towards higher and higher frequencies.

As in the case of almost periodic CA, there are macroscopic quantities which are invariant under the Vlasov flow. Examples are the energy $H[f,g]$, the momentum $M[g]$, or the angular momentum $M[f \wedge g]$.

We can also find almost periodic Bernstein–Green–Kruskal modes [10], which are spatial almost periodic equilibrium measures for the actual Vlasov PDE. These well known solutions are obtained with the separation ansatz

$$P(x,y) = C \exp(-\beta(\frac{y^2}{2} + V \star Q(x))) = S(y)Q(x),$$

where

$$V \star Q(x) = \lim_{n \to \infty} (2n)^{-d} \int_{[-n,n]^d} V(x-x')Q(x')\, dx'.$$

One gets an equilibrium measure if Q solves the integral equation $Q = \exp(-\beta V \star Q(x))$. With $Q = \exp R$, one can find an almost periodic potential V satisfying $\hat{V}(\lambda) = \hat{R}(\lambda)/\widehat{\exp R}(\lambda)$ such that $P(x,y) = S(y)Q(x)$ is an equilibrium solution.

We define the Lyapunov exponent

$$\lambda(\omega) = \limsup_{t \to \infty} t^{-1} \log ||DX_t(\omega)|| \in [0,\infty].$$

Because

$$\begin{aligned}\frac{d}{dt} DX_t = \frac{d}{dt}\begin{pmatrix} Df_t \\ Dg_t \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -M[D^2V(f(\omega)-f(\cdot))] & 0 \end{pmatrix}\begin{pmatrix} Df_t \\ Dg_t \end{pmatrix} \\ &= A(f_t) DX_t\end{aligned}$$

this is the Lyapunov exponent of the finite dimensional cocycle $A(f,g)$ over the flow $X_t = (f_t, g_t)$. If D^2V is globally bounded, then $\lambda(\omega) < \infty$. One can readily check that if the measure m is a finite Dirac measure representing n particles, then this Lyapunov exponent is the classic Lyapunov exponent of a test particle forming together with the n particles a restricted $(n+1)$-body problem.

The dynamical entropy $\lambda = \limsup_{t \to \infty} t^{-1} M[\log ||DX_t(\omega)||] \in [0,\infty]$ is defined because $\omega \mapsto ||DX_t(\omega)||$ is spatial almost periodic for any t. The number λ is a measure for the growth rate of complexity of the almost periodic fluid. If $\lambda > 0$, the almost periodic Dirichlet integral $M[||DX_t||^2]$ grows exponentially and the characteristic flow X_t is not recurrent even though the dynamics are reversible. If $\lambda(\omega) > 0$ for almost all ω, one expects that P_t can converge in a weak sense to an equilibrium solution of the Vlasov equations.

Acknowledgments

E. Reed acknowledges a SURF research fellowship at Caltech during the summer of 1995.

References

[1] W. Brown and K. Hepp, "The Vlasov Dynamics and Its Fluctuations in the $1/n$ Limit of Interacting Classical Particles," *Communications in Mathematical Physics*, **56** (1977) 101–113.

[2] J. Hardy, O. de Pazzis, and Y. Pomeau, "Molecular Dynamics of a Classical Lattice Gas: Transport Properties and Time Correlation Functions," *Physical Reviews A*, **13** (1976) 1949–1961.

[3] J. Hardy and Y. Pomeau, "Thermodynamics and Hydrodynamics for a Modeled Fluid," *Journal of Mathematical Physics*, **13** (1972) 1042–1051.

[4] A. Hof and O. Knill, "Cellular Automata with Almost Periodic Initial Conditions," *Nonlinearity*, **8** (1995) 477–491.

[5] O.E. Lanford III, "Time Dependent Phenomena in Statistical Mechanics," In *Mathematical Problems in Theoretical Physics*, **Volume 116** of *Lecture Notes in Physics* (Springer, Berlin-New York, 1980). Proceedings International Conference in Mathematical Physics, Lausanne, 1979.

[6] A. Jakobsen and O. Knill, "Iteration of the Coupled Map Lattice Construction," *Physics Letters A*, **205** (1995) 179–183.

[7] O. Knill, "Isospectral Deformations of Random Jacobi Operators," *Communications in Mathematical Physics*, **151** (1993) 403–426.

[8] O. Knill, "Nonlinear Dynamics from the Wilson Lagrangian," *Journal of Physics A: Mathematical and General*, **29** (1996) L595–L600.

[9] J. Milnor, "On the Entropy Geometry of Cellular Automata," *Complex Systems*, **2** (1988) 357–386.

[10] H. Neunzert, "An Introduction to the Nonlinear Boltzmann–Vlasov Equation," in *Kinetic Theories and the Boltzmann Equation*, **Volume 1048** of *Lectures Notes in Mathematics* (Springer, Berlin, Heidelberg, 1984).

[11] E. Reed, "Lattice Gas Cellular Automata with Almost Periodic Initial Conditions," Caltech Undergraduate Journal, 1996.

[12] R.Johnson and J.Moser, "The Rotation Number for Almost Periodic Potentials," *Communications in Mathematical Physics*, **84** (1982) 403–438.

[13] H. Spohn, "Large Scale Dynamics of Interacting Particles," *Texts and Monographs in Physics* (Springer-Verlag, New York, 1991).

[14] S. Wolfram, "Theory and Applications of Cellular Automata," (World Scientific, 1986).

[15] S. Wolfram, "Thermodynamics and Hydrodynamics of Cellular Automata," *Statistical Mechanics of Cellular Automata*, 1986. Reprinted in [14].