Complex Systems **8** (1994) 127–150

Cellular Automata in the Triangular Tessellation

Carter Bays
Department of Computer Science,
University of South Carolina, Columbia, SC 29208, USA

For the discussion below the following definitions are helpful. A *semitotalistic CA rule* is a rule for a cellular automaton (CA) where (a) we tally the living neighbors to a cell without regard to their orientation with respect to that cell, and (b) the rule applied to a cell may depend upon its current status. A *lifelike rule* (LFR rule) is a semitotalistic CA rule where (1) cells have exactly two states (alive or dead); (2) the rule giving the state of a cell for the next generation depends exactly upon (a) its state this generation and (b) the total count of the number of live neighbor cells; and (3) when tallying neighbors of a cell, we consider exactly those neighboring cells that touch the cell in question. An LFR rule is written $E_lE_hF_lF_h$, where E_lE_h (the "environment" rule) give the lower and upper bounds for the tally of live neighbors of a currently live cell C so that C remains alive, and F_lF_h (the "fertility" rule) give the lower and upper bounds for the tally of live neighbors required for a currently dead cell to come to life. For an LFR rule to specify a game of Life we impose two further conditions: (A) there must exist at least one glider (translating oscillator) that is discoverable with probability one by starting with finite random initial configurations (sometimes called "random primordial soup"), and (B) the probability is zero that a finite random initial configuration leads to unbounded growth. Note that this second condition does not eliminate the possibility that some unusual highly organized configuration can be constructed where the growth is unbounded. Note also that we may be able to construct some extremely complex configuration that translates; however, if the possibility of discovering this with a random experiment is zero then condition (1) has not been met. We shall call LFR rules that satisfy (A) and (B) *GL* ("Game of Life") rules; they will usually be written "Life $E_lE_hF_lF_h$" (otherwise we simply write "rule $E_lE_hF_lF_h$"). To date there has been only one *GL* rule discovered in two dimensions: that is of course the famous Conway game, Life 2333, which exists on a two-dimensional grid of square cells, where each cell has 8 touching neighbors.

An entire new universe unfolds when we consider the grid (also called a *tessellation* or *tiling*) of equilateral triangles. Here each cell has 12 touching neighbors: three on the edges and nine on the vertices (see Figure 1). We

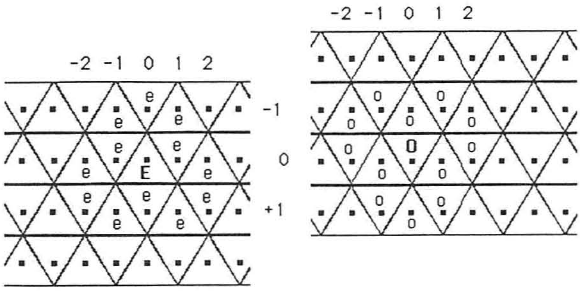

Figure 1: Each cell in △ has 12 touching neighbors. There are two types of cells: E and O cells. The neighbors for each are shown as e or o, respectively. The numbers give the relative locations of the neighbors if the cell whose neighborhood we are evaluating is at location $(0, 0)$. The square dots specify cell locations as simulated in a two-dimensional array.

should first take note of the far greater number of LFR rules possible in the triangular tessellation (hereafter designated △) than in the square grid (which we will denote □). The environment rule can be as small as (0 0) and can encompass all the possibilities up to (12 12): $13 + 12 + 11 + \cdots = 91$ possible values plus the rule "no cell remains alive," which can be written $(- \; - \; F_l \, F_h)$. The fertility rule can have similar values, yielding a possible total of $92 \times 92 = 8464$ LFR rules. Before proceeding, we should note some facts about our triangular universe.

Theorem 1. *Any LFR rule where* $F_l \leq 2$ *leads to unbounded growth.*

Proof. The proof is obvious: simply look at Figure 1 and note that once we place two cells adjacent to each other, growth will proceed regardless of the values for the environment rule. ∎

Theorem 2. *Any LFR rule where* $F_l \geq 6$ *cannot grow without bounds. (We call these rules bounded rules.)*

Proof. Again, simply examine Figure 1 or 2 and note that along the outside of any straight or convex border, no currently dead cell can possibly touch more than 5 cells. ∎

(Similar theorems for □ yield values of $F_l \leq 2$ causing unbounded growth and $F_l \geq 4$ causing bounded growth; for the hexagonal tessellation the values are 1 and 3, respectively.)

Behavior for a typical bounded rule in △, rule 1868, is shown at the left in Figure 2. For the patterns depicted here, all activity is confined to the area within the border. For these rules most patterns either disintegrate totally, stabilize (oscillators of period 1), or evolve into extremely high-period oscillators that are usually contained within convex enclosures. The histogram

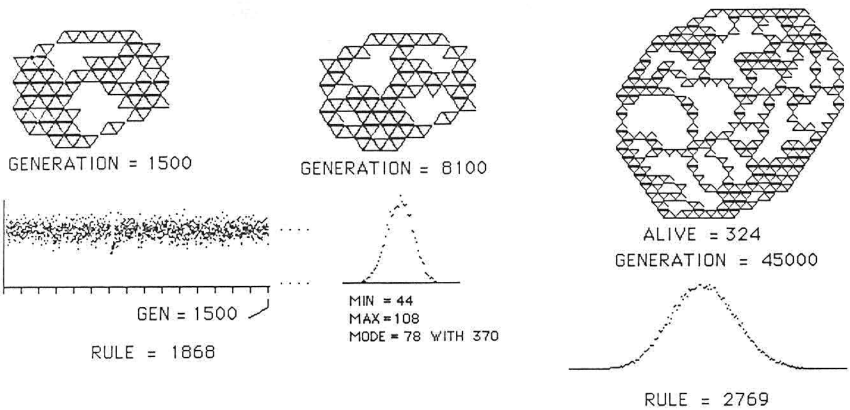

Figure 2: Bounded rules with a large range between E_l and E_h and between F_l and F_h can produce configurations such as these. The histograms show the population tallied after each generation. The curves depict the distribution of population counts after the number of generations indicated. Not unexpectedly the distribution appears to be normal.

at the left depicts the population at each generation out to 1500 generations (the period was not determined). The curve in the middle gives the distribution of the total live population at each generation for a total of 8100 generations.

At the right of Figure 2 we see a somewhat larger pattern, along with the distribution of the count of live cells at each generation after 45,000 generations. Here rule 2769 was used. The vertical scale in Figure 2 is not the same for the two distribution plots.

A measure of the great length of the period for such structures is given as follows. First we are given a boundary (its size is unimportant) plus an interior region of n cells that remain in turmoil; we shall let k generations pass. Note that there are 2^n possible patterns. For our discussion we can also assert that $100 < n \ll k \ll 2^n$. We simplify by assuming that each pattern from generation to generation is independent of the previous pattern. (Our simplifying assumption is not true, but the apparent normal distribution of the count of live cells indicates that our assumption is not an invalid model.) Then the probability P that we will not have created a previously encountered pattern by the $(k+1)$st generation is

$$[2^n \cdot (2^n - 1) \cdot \cdots \cdot (2^n - k)]/2^{(k+1)n}$$

Expanding and dropping the unimportant terms gives approximately

$$[2^{(k+1)n} - (1/2)k(k+1)2^{kn}]/2^{(k+1)n} = 1 - (k^2/2^{n+1}),$$

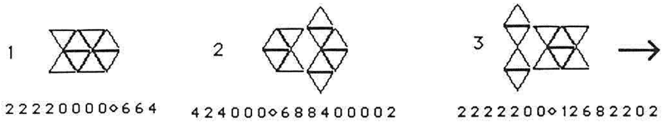

Figure 3: The glider for Life 4644 is the most easily discovered glider for any of the GL rules. It has a period of three, after which it moves one cell to the right. There are six orientations. The numbers under each phase, and similar numbers on the other glider figures, give the signature (see [1]).

hence the probability that we will have encountered a previous pattern by the $(k+1)$st generation is simply $k^2/2^{n+1}$. Thus, for example, k could be as large as, say, $2^{n/4}$; with $n = 300$, the probability of entering a cycle is extremely small.

One of the most fascinating characteristics of LFR rules in Δ is that there are (at least) six GL rules. This is of interest since only one two-dimensional GL rule (Conway's rule) is currently known. These six two-dimensional GL rules are (in the order discovered) Life 4644, 3445, 4546, 2346, 3446, and 2345. It is interesting to note that all GL rules discovered to date contain $E_l E_h F_l F_h$ numbers that lie within the range of values specified by Theorems 1 and 2. Of further interest is that, even though the rules share many common environment and fertility ranges, each behaves in its own distinct way. With one exception each sports at least one glider unique to the rule, and a host of small oscillators all of which are easily discoverable by employing random "primordial soup" experiments (see [1]). Figures 3 through 7 depict the gliders for these various GL rules, and Figures 9 through 14 illustrate some of the oscillators. The richest rules in terms of easily discoverable oscillators appear to be Life 4644 and 4546. All of the experiments to produce oscillators were run on a Macintosh until new oscillators were not being readily produced. A typical run consisted of about 1000 experiments, where each experiment started with a small random pattern. Signatures were utilized (see [1]) to help weed out duplicate patterns. Note that Life 2345 and 2346 share the same glider and have several oscillators in common. The same is not true for Life 3445 and 3446.

In Figure 15 we have shown the rate at which identical random experiments for each GL rule lead to stable configurations (i.e., all patterns have converged to oscillators with a finite period ≥ 1). Rule 1246 also sports a glider (see Figure 8), but unfortunately leads to unbounded growth (see Figure 16); hence it is not a GL rule.

The left of Figure 17 shows the growth rate of Life 2333 by comparison (here, of course, the grid is square rather than triangular). Note that experiments under Conway's rule yield much more residue than any of the Δ GL rules. Furthermore, Life 2333 requires more time to settle into a stable configuration than any of the Δ GL rules. The implication here is that devices such as "glider guns" (devices that spew forth endless supplies of gliders) will

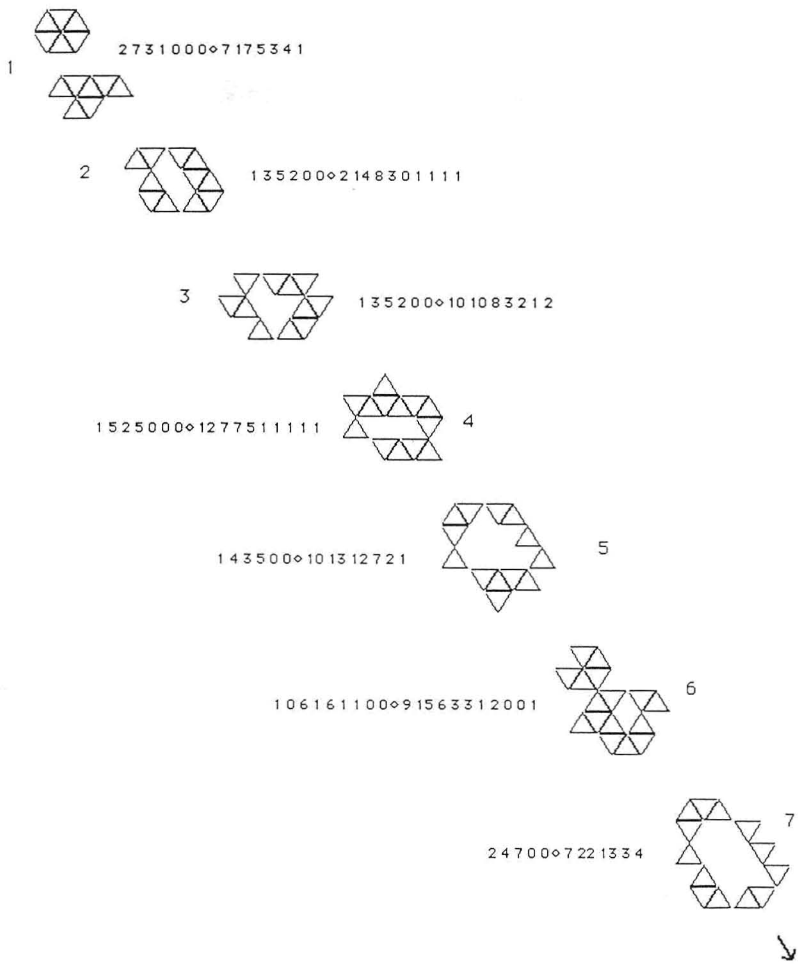

Figure 4: The glider for Life 3445 (and other gliders) are harder to discover than the glider in Figure 3. This glider has an unusual period of 7 and moves one cell per cycle in the direction shown. It has 12 possible orientations. No other oscillator for any GL rule has been discovered whose period is seven.

be hard to discover in Δ. Perhaps this difficulty is overcome somewhat by the large number of GL rules available.

At the right of Figure 17 we see the growth behavior of the very interesting Δ rule 2333. Growth for this rule decays very slowly, going through wild gyrations. All random configurations tested did eventually stabilize. The

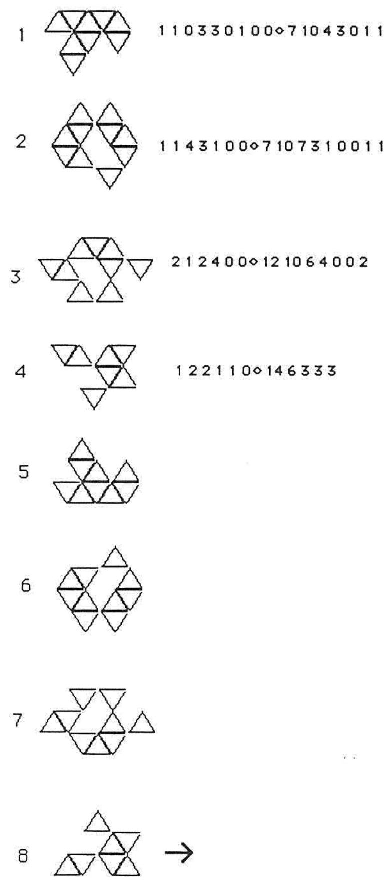

Figure 5: The period-eight Life 4546 glider moves one cell per cycle in the direction shown. Even though it is asymmetric, there are only six orientations, since the second half of the period is a reflection of the first.

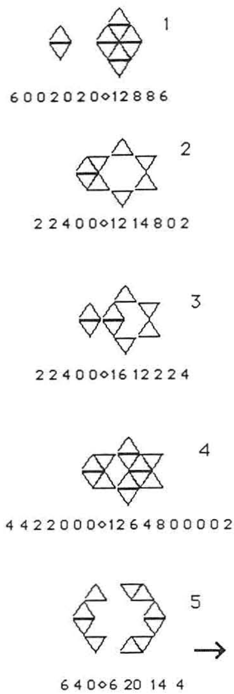

Figure 6: The symmetric Life 2345 and 2346 glider has a period of five. This glider, though somewhat similar to the Life 4644 glider, is more difficult to discover. It has six orientations and moves one cell per cycle in the indicated direction.

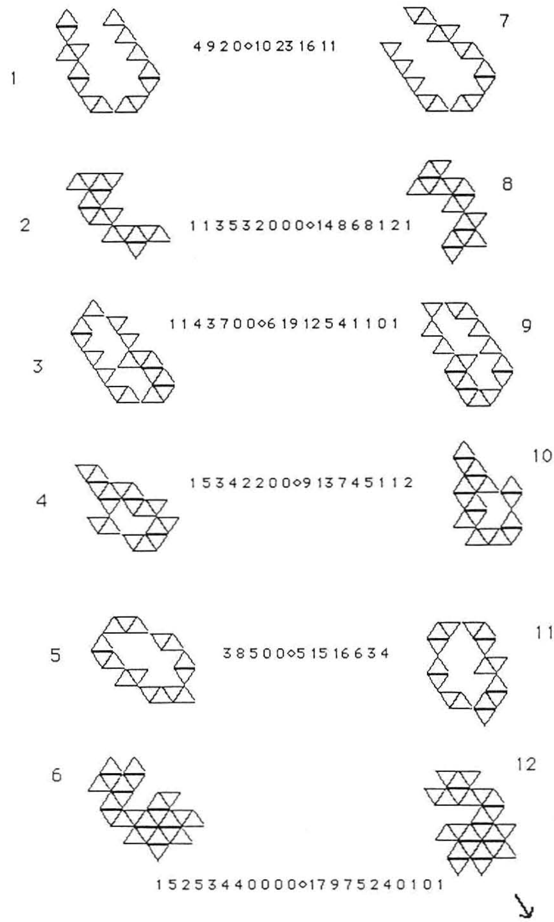

Figure 7: Here again we have an asymmetric glider where the second half of the cycle is a reflection of the first. This Life 3446 glider has a period of 12 and moves two cells per cycle.

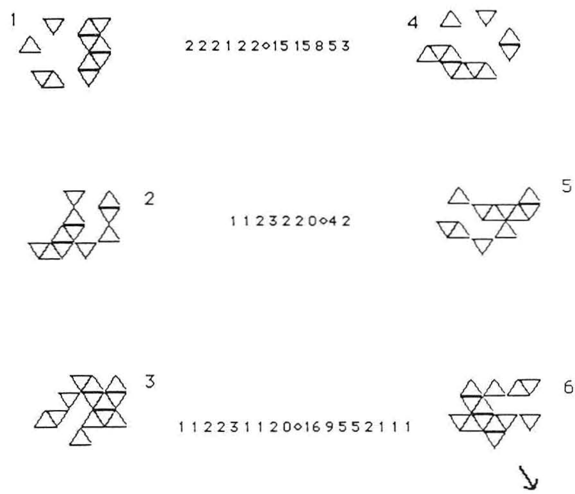

Figure 8: Rule 1246 sports a period-six glider that moves one cell per cycle in the indicated direction. Since 1246 allows unbounded growth, it does not qualify as a GL rule (see Figure 16).

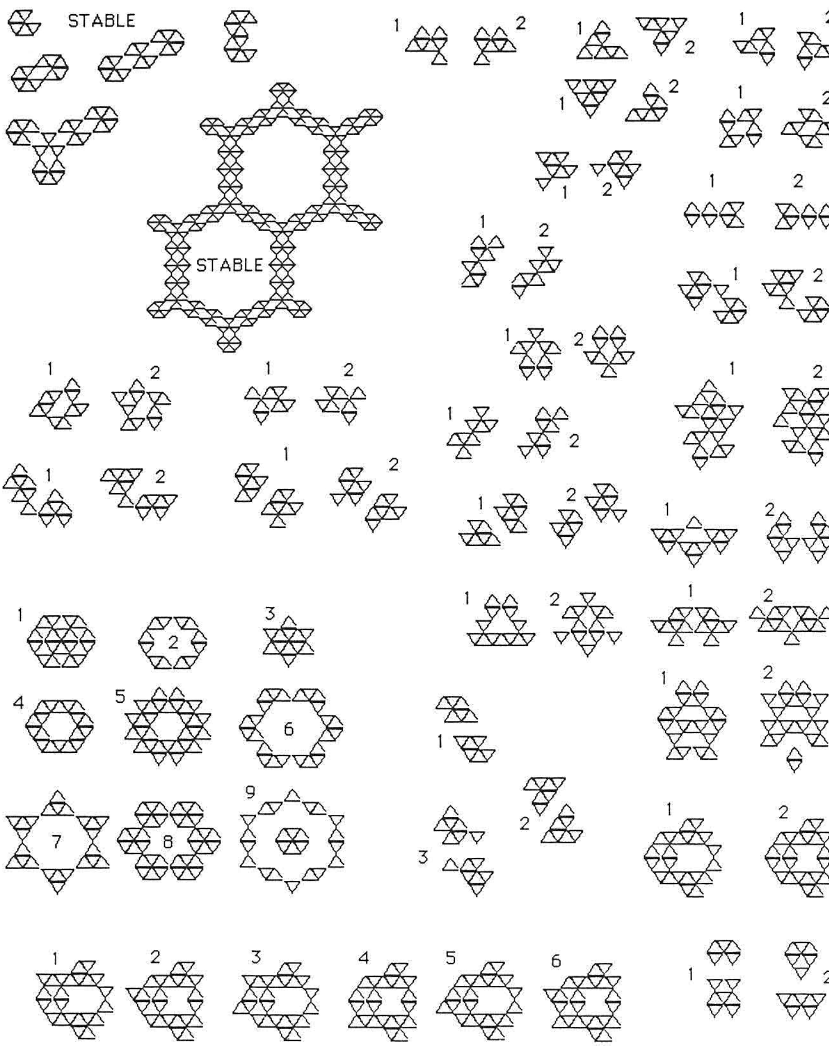

Figure 9: Here are the most common oscillators for Life 4644. The stable structures at the upper left can be made arbitrarily large.

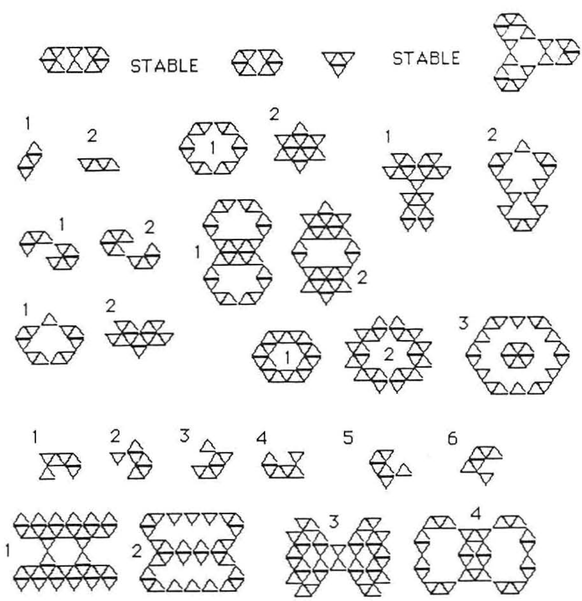

Figure 10: Depicted are the most common Life 3445 oscillators. These varieties are all that condense from experiments after a several hour run on a Macintosh.

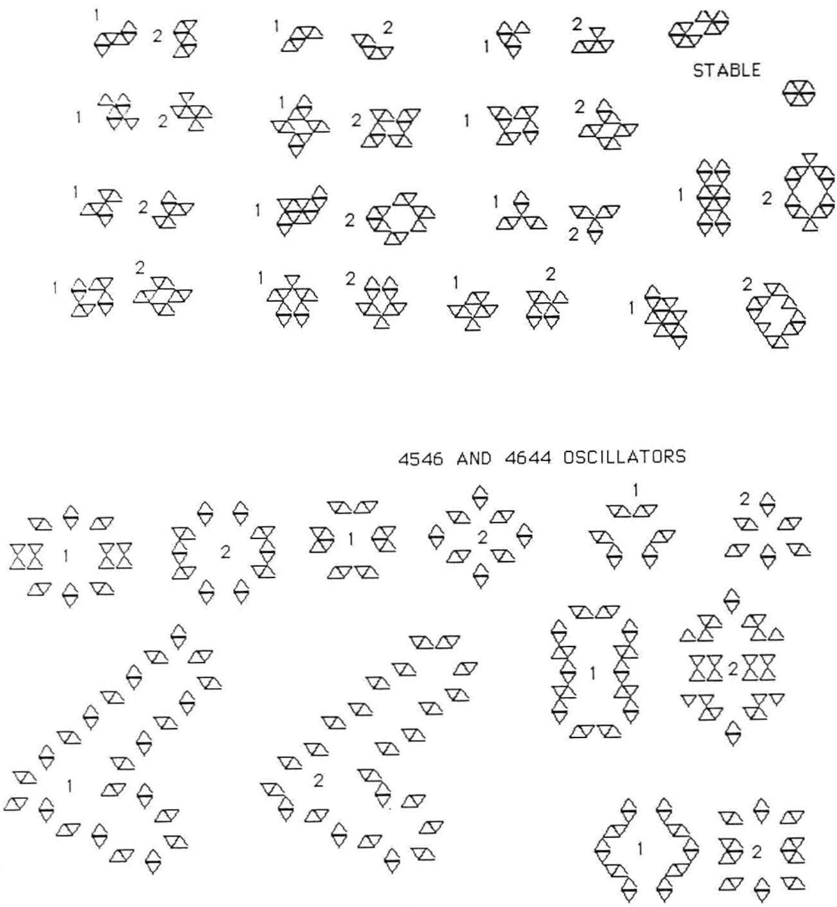

Figure 11: The most prolific GL rule appears to be Life 4546. Here are the most common oscillators with period ≤ 2. At the bottom are shown oscillators that occur for both Life 4546 and 4644. Note that these forms can be arbitrarily large.

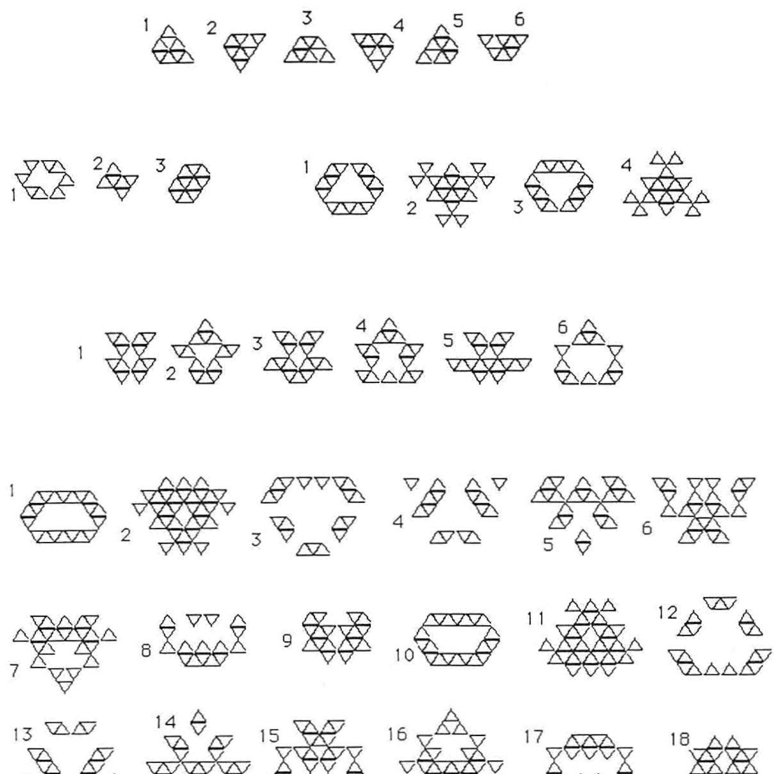

Figure 12: These are the common oscillators for Life 4546 where the period exceeds two.

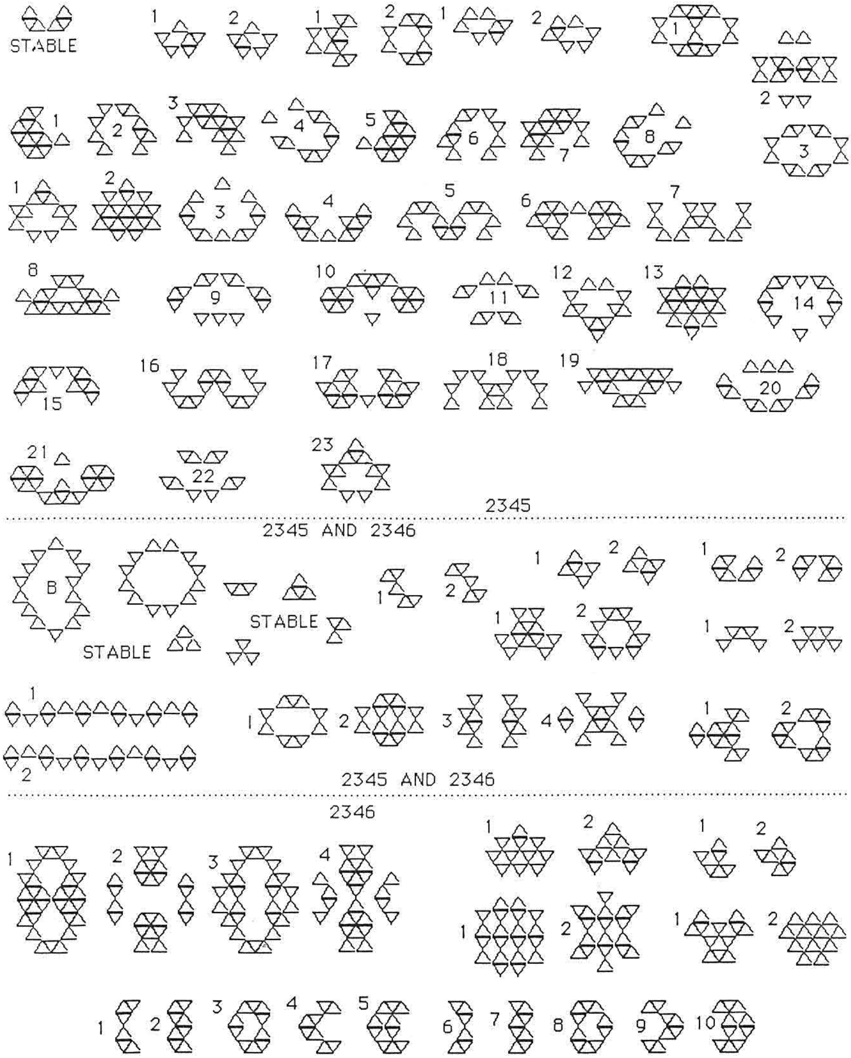

Figure 13: Here are the common Life 2345 and 2346 oscillators. Although Life 2346, 3446, and 4546 share the same fertility rule, the three rules have few oscillators in common; Life 2345 and 2346 do, however. At the top we have oscillators that occur in Life 2345 but not 2346. At the bottom we have forms that appear in Life 2346 but not 2345. Between the dotted lines are oscillators common to both rules. The period-22 oscillator illustrated at the top is the longest period GL oscillator discovered to date by primordial soup experiments.

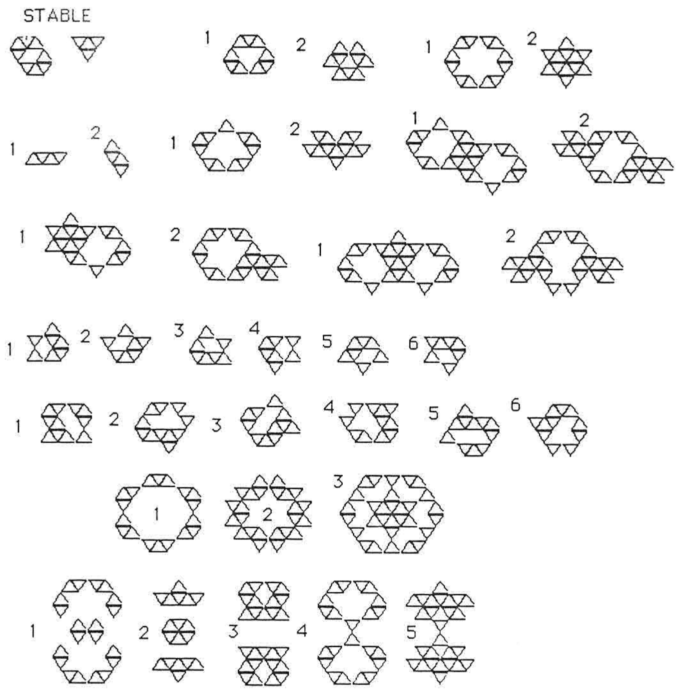

Figure 14: Random experiments produce these forms for Life 3446. It is likely that if a great many more experiments were run, then all GL rules would reveal more such forms. Although some oscillators occur for both Life 3445 and 3446, there appears to be not nearly the overlap as that between 2345 and 2346.

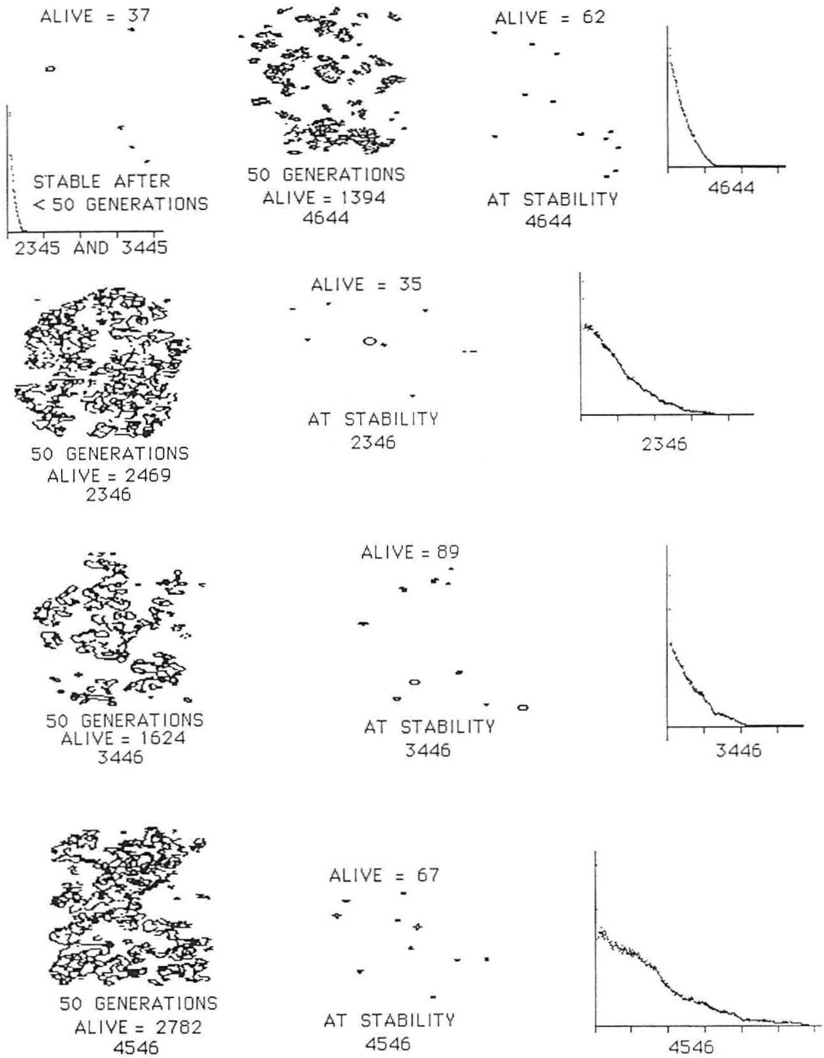

Figure 15: The plots illustrate the relative rates at which experiments with random starting patterns will stabilize. In each case, 40% of a 100 × 100 grid was filled with live cells. Although individual experiments produce somewhat different results, the GL rules can be ordered according to their rates of stabilization, from fastest to slowest, as: 2345 and 3445, 4644, 3446, 2346, 4546. For Figures 15–20 the intervals in the plots each represent 100 generations. (For each of the plots in Figures 15–20 the overall size of the grid was 400 × 400, but the starting size for the random blob in the center varies.)

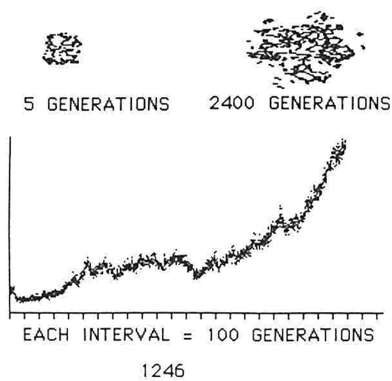

Figure 16: The rule 1246 supports a glider but is not a GL rule because growth is expansive.

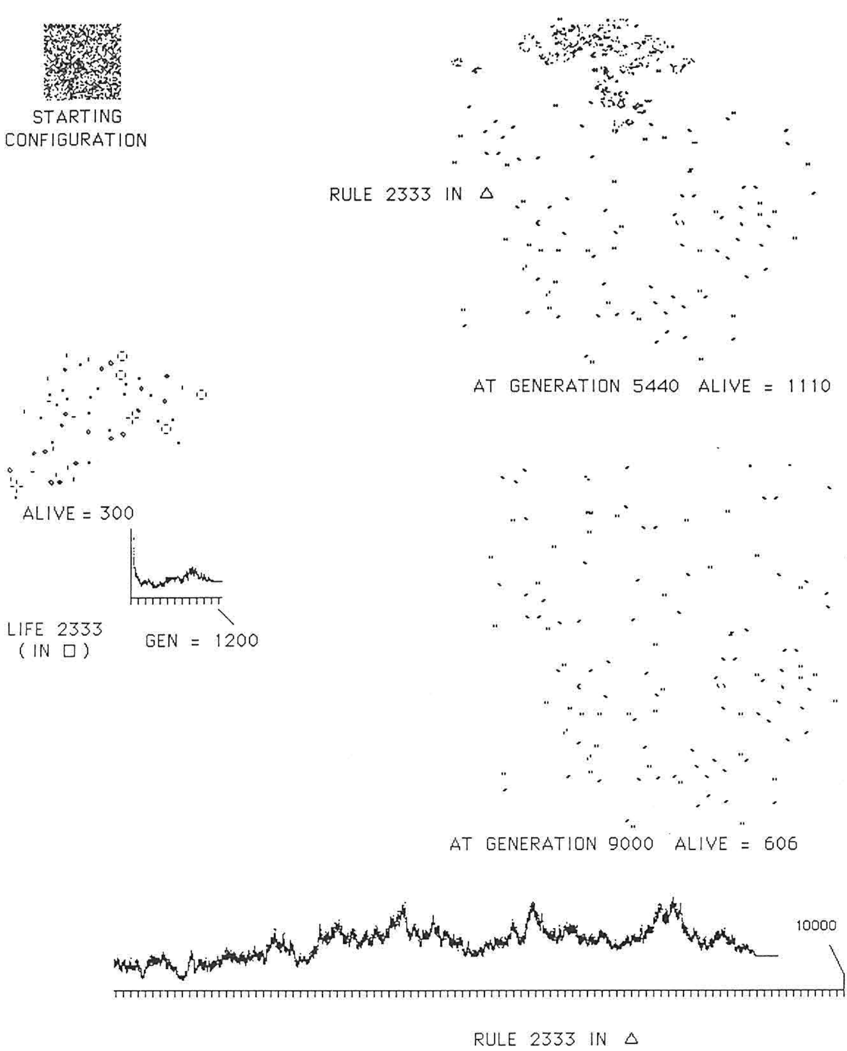

Figure 17: Here we see for comparison (at the left) Conway's rule, Life 2333 (in □), which requires much more time to stabilize and leaves behind much more residue than any of the GL rules in △. The △ rule 2333 appears to have bounded growth and stabilizes even more slowly than Life 2333. So far no glider has been discovered for this rule. The starting configuration for both rules is shown at the upper left.

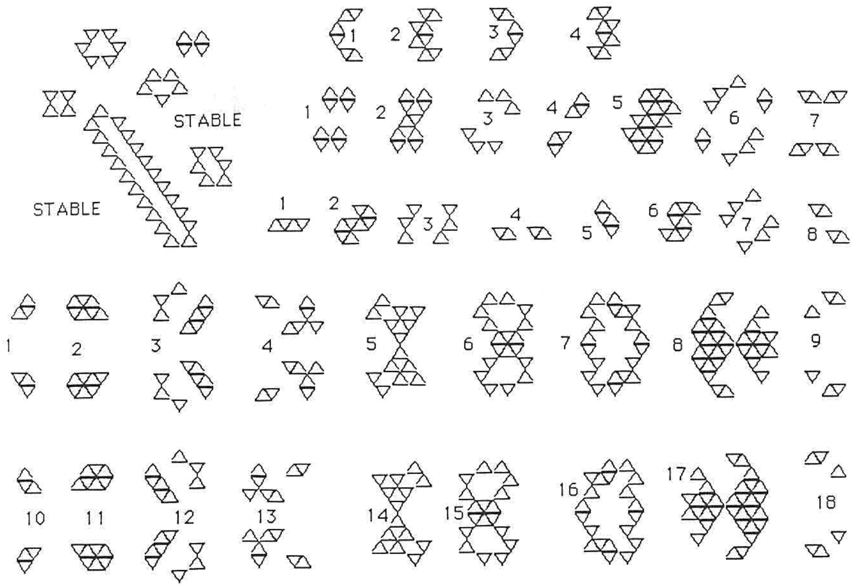

Figure 18: Here are a few of the oscillators for rule 2333. After a considerable search, no period-two oscillators were found. Of all the rules whose oscillators were investigated, rule 2333 was the only rule with this characteristic.

one lacking feature is the failure of 2333 so far to produce a glider. Other interesting features of rule 2333 include the fact that no period-two oscillators have yet been discovered, and that among rules 0133, 1233, 2333, and 3433, rule 2333 is the only rule where growth is bounded (see Figures 17 and 19). The oscillators that were discovered for rule 2333 gyrate wildly, leading one to hope that some glider might yet be found. Even if this is the case, this rule would probably not yield the rich constructs of Life 2333 because it is the relative abundance of the glider that gives Conway's rule its allure.

In Figure 20 we note the effect of altering the Life 3445 rule slightly, changing 3445 to 3544. No glider has yet been discovered for 3544, but perhaps more surprising is the fact that rule 3544 requires much more time to settle down than does 3445. (Coincidentally, for plots in Figures 17 and 20, rules 2333 and 3544 both required about the same number of generations to stabilize.)

In each case in Figure 15 the configuration started out as a random block, 100×100, filled 40% with live cells. For Figures 17 and 20 the starting block was 25×25, and for Figures 16 and 19 it was 10×10. The overall grid size was 400×400 for each of the plots.

Figure 21 depicts some of the oscillators for rule 3544. It should be pointed out that a great many rules have small unique oscillators, regardless

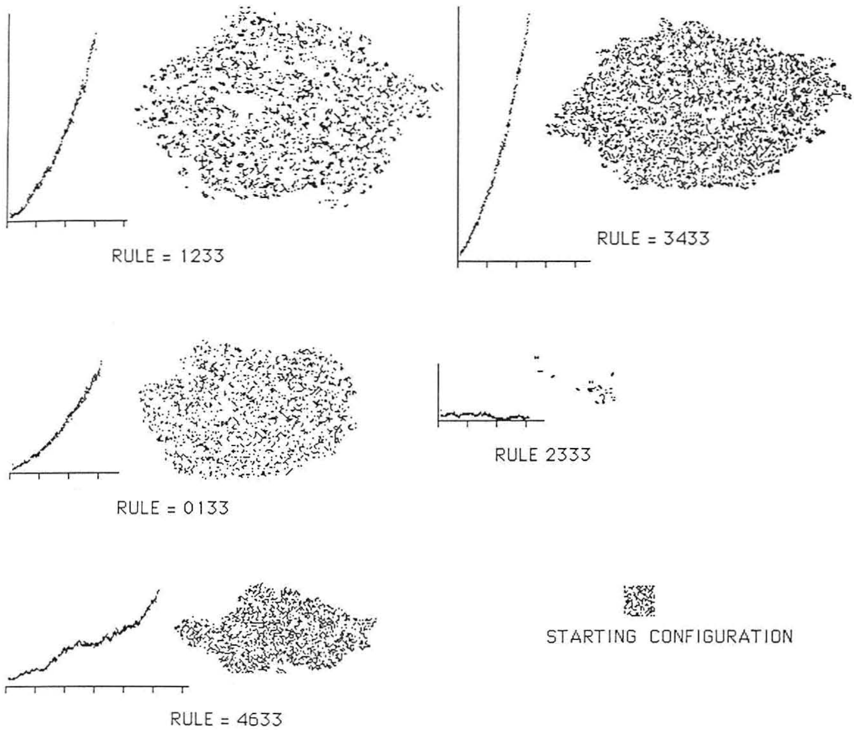

Figure 19: Another interesting fact about rule 2333 is that it is the only rule among 0133, 1233, 2333, and 3433 that does not allow unbounded growth. A typical random 10×10 starting configuration leads to the patterns shown. The tiny plot for rule 2333 is at the same scale as the others. It was not run out to stabilization, which did eventually occur. (See also Figure 17.) Growth for rule 4533 decayed very quickly; rule 4633 led to unbounded growth.

of whether they are GL rules. In our search for GL rules we investigated several LFR rules, many of which satisfied criterion (B) but not (A); that is, no glider has yet been discovered. At least 10^6 glider search experiments were performed on each of the following rules: 1245, 4533, 3444, 4544, 3544, 4545, 2233, 2333, 3433, 3333, 2444, 2445, 4446, 4645, 5646, 1246, 2345, 5746, and 0145. Many rules not listed here were abandoned once it was determined that they supported unbounded growth. Although the rules listed above seemed to the author to be the most likely candidates for supporting gliders, there remain many rules to examine.

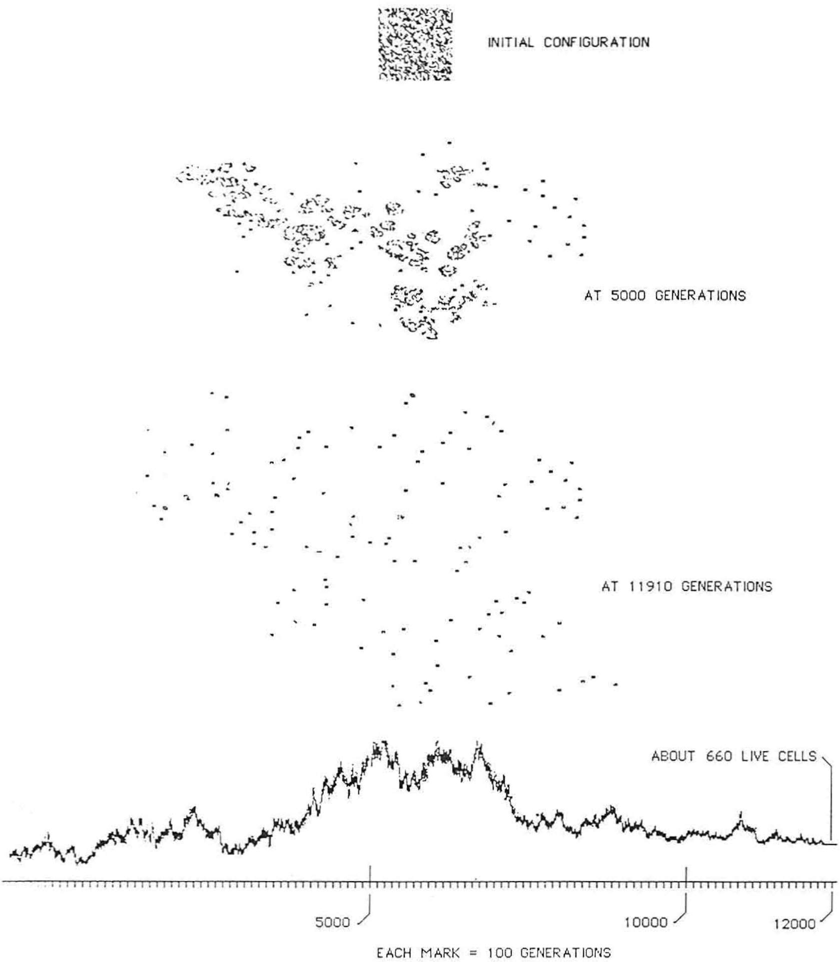

Figure 20: Every experiment run with rule 3544 went through violent changes in population but eventually stabilized. This experiment stabilized at 11,910 generations.

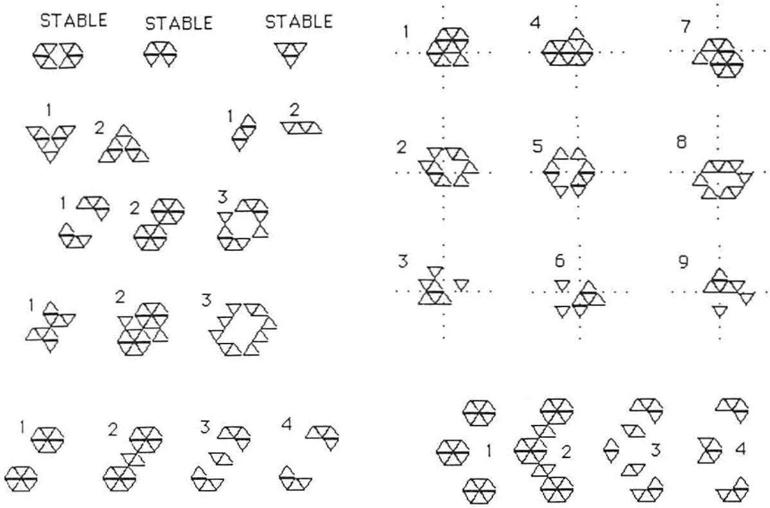

Figure 21: Here are some oscillators for rule 3544, which is probably not a GL rule since no glider has yet been discovered. The form at the right tries, but does not move very far, revolving about the point indicated by the intersection of the dotted lines. A great many LFR but non-GL rules in Δ support small oscillators that are unique to the rule.

Programming method

Perhaps the most straightforward way to implement Δ is to utilize the configuration shown in Figure 1. Here each cell contains a dot positioned in such a way as to define a rectangular array. Note that we have "even" cells (triangles resting on their bases) and "odd" cells. We then utilize a two-dimensional array and "neighborhood" templates. For a cell at relative location $(0,0)$ the template specifies the neighbors. For the even (E) cells the template is $(-1,-1;\ -1,0;\ -1,1;\ 0,-2;\ 0,-1;\ 0,1;\ 0,2;\ 1,-2;\ 1,-1;\ 1,0;\ 1,1;\ 1,2)$, and for odd (O) cells it is $(-1,-2;\ -1,-1;\ -1,0;\ -1,1;\ -1,2;\ 0,-2;\ 0,-1;\ 0,1;\ 0,2;\ 1,-1;\ 1,0;\ 1,2)$. We determine whether a cell is even or odd by finding $(I+J) \bmod 2$, where I and J are subscripts for the two-dimensional array. Depending upon what kind of templates we place in our program, we can thus utilize the same program to explore cellular automata in many regular tessellations—for example, the hexagonal tessellation, the Cairo tessellation (a tiling of identical equilateral pentagons), or the square tessellation. We can even investigate tessellations composed of more than one type of polygon (see Figure 22). Here we would require templates for each different polygon type; the proper template(s) would be employed when we investigate the neighborhood for that polygon. Of course we must alter the graphic output to depict the proper cell layout for each tessellation being run.

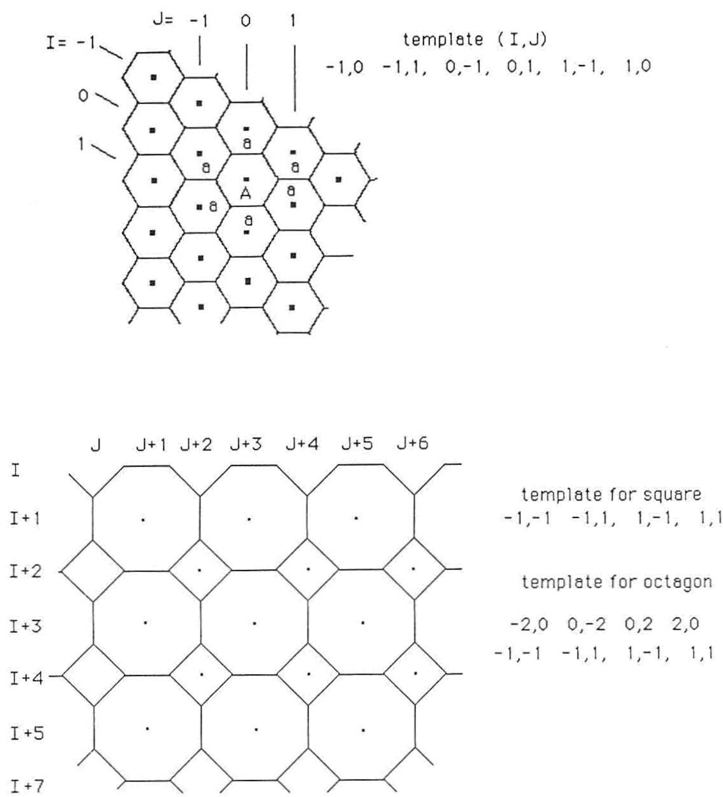

Figure 22: Templates for other tessellations can be easily constructed and simulated with a square array. When more than one type of polygon is present we can easily find which we are dealing with by looking at $(I + J) \bmod n$, where n is usually two or some other small number. Of course the graphic output for each polygon or orientation must be set up properly.

With relatively small populations in large universes we gain speed if at each generation we only examine cells that will die (i.e., live cells whose neighbor count lies outside the environment range), or cells that will come to life (dead cells whose neighbor count lies within the fertility range). Hence each cell contains a count of the number of live neighbors along with a flag to indicate whether it is dead or alive. We need only store a list of changes as we evaluate each new generation by rapidly scanning over the grid of cells. Then we update the neighbor count for each neighbor of each changed cell.

References

[1] Carter Bays, "Patterns for Simple Cellular Automata in a Universe of Dense Packed Spheres," *Complex Systems*, **1** (1987) 853–875.

[2] Stephen Wolfram, *Theory and Applications of Cellular Automata* (Singapore: World Scientific, 1986).