A Thermodynamic Automaton with Four Temperatures and Three Controls

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Abstract. A Thermodynamic Automaton is a finite set of subsystems interacting according to the laws of non-equilibrium thermodynamics. The thermal flows between subsystems, determining the evolution in the state space, are modulated by bilinear controls that map the state space onto a finite set. The physical motivation for introducing Thermodynamical Automata is the problem of obtaining a thermostat ambient with zero energy consumption. The mathematical context of Thermodynamic Automata is the theory of finite-dimensional dynamical systems: the general distributed parameter problem is reduced to a lumped parameter problem via compartmental modeling. Corresponding to various degrees of complexity of the system, various theoretical models can be developed; a compartmental reduction into four components and three control functions is studied here and is compared with Optimal Control Theory.

1. Introduction

Our program is the study of passive feedback control of the heat flows in a structure subject to time-dependent boundary conditions: the surrounding temperature and the solar radiation flux.

"Passive control" means that the control functions appear in the state equations as bi- or multilinear terms. Multiplicative controls do not imply heat sources or sinks, but instead act on the existing heat flows. Our approach is to apply multiplicative controls, avoiding the controls of the additive kind.

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whose engineering and mathematics are well known. The presence of feedback controls (the controls are functionals of the state variables) generates higher-order nonlinearities.

The model we are going to study is a particular reduction of the following general problem.

Consider a finite domain $\Omega$ embedded in the terrestrial environment $E$, which can be described by at least two fields: the temperature $T_E(r, t)$ (K) and the radiation flux $\phi_{\text{in}}(r, t)$ (W m$^{-2}$). These are the driving terms of the problem and appear in the boundary conditions. In the present study they are deterministic periodic functions; only the period $\tau$ of terrestrial rotation is considered while the annual period is neglected.

Inside the domain the physical system is governed by the field equations of non-equilibrium thermodynamics. Our aim is to control the temperature field $T(r, t)$ in a subdomain $\Omega_{\text{core}} \subset \Omega$ with minimum mean-square deviation of $T$ from a desired value $T_b$ (the index $b$ standing for "best").

There are many (infinite) ways according to which a control field $c(r, t)$ could appear in the field equations in order to satisfy the above-mentioned requirement: we restrict to zero energy control systems, namely systems that do not require additive energy sources or sinks (additive controls). The restriction to zero energy systems entails that the control field enters the equations as a multiplicative term.

In section 2 we proceed from these general statements to a particular lumped parameter model by means of a compartmental reduction. In relation to this, there are two possible approaches: either (1) working with a compartmental subdivision rich enough to give a good approximation to the field equations of a given experimental object, or (2) working with a model as simple as possible that contains the ideal properties of the problem. Our approach is the second one, which we find more thought-provoking. An analogy is the simplicity of the ideal Carnot cycle, which is quite distant from a real thermal engine. Our goal here is, in a manner of speaking, to study à la Carnot the thermal control of a structure subject to geophysical inputs. Therefore we do not claim high fidelity to experimental situations, but rather we strive for the minimum number of equations that is able at the same time to contain the essential features of the problem.

From the standpoint of Optimal Control Theory [1], an optimally controlled, non-autonomous, bilinear dynamical system with no energy cost is solved both formally by means of the Bang-Bang Principle and numerically by means of Dynamic Programming (section 4). This approach requires time-consuming analytical and/or numerical techniques.

In section 5 we propose a control strategy that closes the feedback loop: the resulting controlled system is a Thermodynamic Automaton (ThA), which is able to handle any unknown driving term. We show that Dynamic Programming and the ThA strategy give essentially the same results, and that therefore the ThA is preferable because of its simplicity.

Finally, the ThA is studied in section 6 as an entropy flux manipulator.
2. The compartmental reduction

The original domain $\Omega$ is decomposed into 4 compartments, in each of which the space dependence is lost. Therefore the temperature field $T(r, t)$ is reduced to the set of 4 functions $T_i(t)$ (see figure 1); each distributed parameter $\lambda(r)$, for example mass density or specific heat, is reduced to the set of 4 lumped parameters $\lambda_i$; and the control field $c(r, t)$ is reduced to the set of 3 control functions $c_{ij}(t)$ represented by a switch in figure 1.

Our case study is therefore the system $Z(4, 3)$: Zero Energy Thermodynamic Automaton with 4 compartments (temperatures $T_i$) and 3 controls $c_{ij}$. The controls are for the time being considered as time-varying coefficients whose explicit expression shall be assigned in sections 4 and 5.

The control requirement affects compartment 1 only, which represents $\Omega_{\text{core}}$:

$$J \equiv \frac{1}{\tau} \int_{t-\tau}^{t} \left[ T_1(t') - T_b \right]^2 dt' \longrightarrow \text{minimum.} \quad (2.1)$$

We refer also to figure 2, which is a realistic configuration of the abstract scheme of figure 1. The control functions $c_{ij}(t)$ are represented with a valve symbol $\otimes$. Compartment 1 represents a dwelling space. Compartment 4
Figure 2: A possible physical realization of the compartmental model reduction. Compartments 1 (the core) and 3 are constituted of air, compartment 2 is the iron structure of the system, and compartment 4 is the solar radiation receiver.

represents the solar radiation receiver, and 3 represents the medium that exchanges heat between 4, E, and compartment 2. Compartment 2 represents the structure of the system in the sense that, if $m_i$ is the mass of $i$, then $m_2 \gg m_1 \simeq m_3 \simeq m_4$.

Figures 1 and 2 show that the radiation fluxes involve only compartment 4 because compartment 3 (e.g., air) and the boundary surface $S_{3E}$ (e.g., glass) are transparent.

The uncontrolled conductive-convective heat flow between $i$ and $j$ is approximated by the phenomenological expression (in W)

$$q_{ij}(t) = \alpha_{ij} S_{ij} [T_i(t) - T_j(t)],$$

where $S_{ij}$ denotes the area of the uncontrolled interface between compartments $i$ and $j$, and $\alpha_{ij}$ (in W m$^{-2}$ K$^{-1}$) is the relative overall heat transfer coefficient (see appendix B).

Similarly the controlled heat flow between $i$ and $j$ through $\tilde{S}_{ij}$ is

$$\tilde{q}_{ij}(t) = \tilde{\alpha}_{ij} \tilde{S}_{ij} c_{ij}(t) [T_i(t) - T_j(t)],$$

where the $\tilde{\cdot}$ marks the presence of the control $c_{ij}$ as a dimensionless factor. The total heat flow through the total $(i, j)$ interface $S^*_{ij} \equiv S_{ij} \cup \tilde{S}_{ij}$ is then

$$q^*_{ij}(t) = q_{ij}(t) + \tilde{q}_{ij}(t) = [\alpha_{ij} S_{ij} + \tilde{\alpha}_{ij} \tilde{S}_{ij} c_{ij}(t)] [T_i(t) - T_j(t)].$$

For sake of simplicity we will use the thermal conductivity coefficients having units W/K:

$$\beta_{ij} \equiv \alpha_{ij} S_{ij}, \quad \tilde{\beta}_{ij} \equiv \tilde{\alpha}_{ij} \tilde{S}_{ij}.$$
so that the heat flow is
\[ q_{ij}^*(t) = \left[ \beta_{ij} + \tilde{\beta}_{ij} c_{ij}(t) \right] [T_i(t) - T_j(t)]. \]  

The choice of the thermal conductivities is such that the controlled part \( \tilde{q}_{ij} \) of \( q_{ij}^* \) has larger weight than the uncontrolled \( \tilde{q}_{ij} \):
\[ \tilde{\beta}_{ij} \gg \beta_{ij}. \]  

The control \( c_{ij}(t) \) takes the meaning of the fraction of opening of \( \tilde{S}_{ij} \) by imposing the range of variability
\[ 0 \leq c_{ij}(t) \leq 1, \]

so that we have
\[
\begin{cases} 
\text{maximum insulation for } c_{ij} = 0, \text{ or } q_{ij}^* = q_{ij}, \\
\text{maximum conduction for } c_{ij} = 1, \text{ or } q_{ij}^* \simeq q_{ij}. 
\end{cases}
\]  

We point out that we have uncontrollable heat flows (see figures 1 and 2) of the kind (2.2). Notice finally that \( q_{ji}(t) = -q_{ij}(t) \) and \( q_{ji}^*(t) = -q_{ij}^*(t) \).
The boundary data are represented by $T_E(t)$ on $S_{1E}$ and $S_{3E}$ and $\phi^{in}(t)$ on $S_{4E}$ (see figures 1 and 2). The heat flow between 1 and E through $S_{1E}$ is given by a term $q_{1E}(t)$ as in equation (2.2); the problem of modeling the heat flow between 3 and E is tackled in the next section. The net radiative exchange of heat (in W) is

$$S_{4E} [\phi^{in}(t) - \phi^{out}(t)] = S_{4E} [\phi^{in}(t) - \varepsilon \sigma T_4^4(t)],$$

where $\sigma$ is the Stefan-Boltzmann constant and the factor $\varepsilon$ is a dimensionless parameter simulating the presence of a selective glass surface with $0 \leq \varepsilon \leq 1$ (in particular, we define $\varepsilon = 0.8$ throughout the present paper).

Figure 3 gives the time dependence of $T_E$ and $\phi^{in}$ (see also appendix A).

3. The model equations

In this section we introduce the system of ordinary differential equations governing the compartmental model. The general expression is

$$\dot{T}(t) = f(T(t), c(t), t),$$

(3.1)

where $T$ is the state vector and $c \equiv (c_{12}, c_{23}, c_{3E})$ is the control vector.

The differential equation governing each compartment is obtained by performing the instantaneous energy flux balance (in W) on the compartment. The internal energy rate of the $i$th compartment is $\mu_i \dot{T}_i$ (W), the quantity $\mu_i$ (in J/K) representing the heat capacity of compartment $i$ (see appendix B). Therefore the most general expression of the $i$th differential equation is of the form

$$\mu_i \dot{T}_i = \sum_{j \neq i} \tilde{q}_{ij} + \sum_{k \neq i} q_{ik} + q_{iE} + S_{iE} [\phi^{in} - \phi^{out}]$$

(3.2)

Figures 1 and 2 assign the effective thermal links expressed by equations (2.2) through (2.4) and equation (2.10).

A crucial problem is how to simulate the extremely rapid equilibration of temperature $T_3$ with the ambient temperature $T_E$ (a thermal bath) when the surface $S_{3E}$ is removed. Depending on how this effect of the control $c_{3E}$ on surface $S_{3E}$ is simulated, there are two different models of the dynamical system: the trilinear [2] and the bilinear models.

The idea underlying the bilinear model is to insert a controlled heat flow $q_{E3}^* = q_{E3} + \tilde{q}_{E3}$ between 3 and E, of the same kind as equation (2.3).

The governing equations are then:

$$\mu_1 \dot{T}_1(t) = [\beta_{12} + \tilde{\beta}_{12} c_{12}(t)] [T_2(t) - T_1(t)] + \beta_{1E} [T_E(t) - T_1(t)]$$

$$\mu_2 \dot{T}_2(t) = [\beta_{12} + \tilde{\beta}_{12} c_{12}(t)] [T_1(t) - T_2(t)]$$

$$+ [\beta_{23} + \tilde{\beta}_{23} c_{23}(t)] [T_3(t) - T_2(t)]$$

$$\mu_3 \dot{T}_3(t) = [\beta_{23} + \tilde{\beta}_{23} c_{23}(t)] [T_2(t) - T_3(t)] + \beta_{34} [T_4(t) - T_3(t)]$$

$$+ [\beta_{3E} + \tilde{\beta}_{3E} c_{3E}(t)] [T_E(t) - T_3(t)]$$

$$\mu_4 \dot{T}_4(t) = S_{4E} [\phi^{in}(t) - \varepsilon \sigma T_4^4(t)] + \beta_{34} [T_3(t) - T_4(t)].$$

(3.3)
The bilinear terms are of the form $c_{jk} T_k$ or $c_{jk} T_j$.

We state without proof [2] that, when $c_{3E} = 1$, $T_a(t) \to T_E(t)$ in the limit $\beta_{3E} \to \infty$.

### 4. The approach of Optimal Control Theory

The problem of satisfying equation (2.1) for the dynamical system (3.3) with controls bounded by equation (2.8) is an optimal control problem that we will solve formally by means of the Pontryagin Minimum Principle and numerically by means of Dynamic Programming.¹

The conditions (2.8) define a set $C$ of admissible controls. The optimal control $c^*$ is that control in $C$ that fulfills equation (2.1); the corresponding optimal state is denoted with $T^*$.

Let us first define the Hamiltonian function as

$$ H(T, c, p, t) = \frac{(T_i - T_b)^2}{\tau} + p \cdot f(T, c, t). $$ (4.1)

The adjoint state vector $p(t)$ adjoins the dynamical equations to the variational requirement and it does not represent any physically meaningful quantity.

The Minimum Principle [1, 3] asserts that a necessary condition for $c^*$ to be optimal is

$$ \begin{cases} \forall t, \forall c \in C, \quad H(T^*, c^*, p^*, t) \geq H(T, c, p, t), \\ \dot{p}_i^*(t) = -\frac{\partial H}{\partial T_i^*} (i = 1, \ldots, 4). \end{cases} $$ (4.2)

The solution of the canonical system given by equations (3.1) and (4.2) is a cumbersome nonlinear, two-point boundary-value problem.

Fortunately for the class of problems we are dealing with we can apply the Bang-Bang Principle [1], which we recall in the following form:

*If the set of equations (3.1) is nonlinear in the state function and bilinear in the state and control functions, and if the control function does not appear in the performance index $J$, then the relationship between $c^*_{ij}$ and its switching function $\varphi_{ij}(T^*, p^*)$ (defined as the opposite of the coefficient of $c_{ij}$ in the Hamiltonian) is

$$ c^*_{ij} = \Theta[\varphi_{ij}(T^*, p^*)] \quad (ij = 12, 23, 3E), $$ (4.3)

where $\Theta$ is the Heaviside unit step function. This kind of optimal control is called bang-bang because the control components assume only the minimum or maximum values allowed by equation (2.8).*
After inspection of the Hamiltonian, whose explicit expression requires easy but cumbersome calculations, we find that \((ij = 12, 23, 3E)\)

\[ c^*_ij(t) = \Theta \left[ \left( \frac{p^*_j}{\mu_j} - \frac{p^*_i}{\mu_i} \right) (T^*_i - T^*_j) \right], \quad (4.4) \]

where the switching function \(\varphi_{ij}\) is the expression within square brackets. Hence \(c^* = (c^*_{12}, c^*_{23}, c^*_{3E})\) assumes at any time one of the \(2^3 = 8\) configurations:

\[
\begin{align*}
(0, 0, 0), & \quad (0, 0, 1), \\
(0, 1, 0), & \quad (0, 1, 1), \\
(1, 0, 0), & \quad (1, 0, 1), \\
(1, 1, 0), & \quad (1, 1, 1).
\end{align*}
\]

The knowledge of the instants of time when a certain control configuration switches to another one—the zeros of the switching functions—comes from the numerical solution of equation (4.2) because \(p\) is not directly measurable.

The optimal control as a function of time represents an open-loop control whose disadvantage is the applicability to the case of explicit driving terms only.

At this point we solve the problem numerically by means of Dynamic Programming (DP)—which is equivalent to the Minimum Principle [1]—using the result (4.5) in order to shorten the calculations. The basis of DP is the Principle of Optimality [1, 3]:

*If a motion is optimal, then each terminal segment of the motion is itself optimal.*

The DP algorithm is the following [1]:

1. The differential equations are transformed into finite-difference equations, therefore if \(\tau\) is the total time interval, \(t = t_0, t_0 + \Delta t, \ldots, t_0 + N \Delta t\), \(N\) being the number of intervals into which \(\tau\) is divided.

2. State and control variables must be subject to constraints:

\[
T^\text{min}_i \leq T_i(t) \leq T^\text{max}_i, \quad c^\text{min}_k \leq c_k(t) \leq c^\text{max}_k \quad (i = 1, \ldots, n; k = 1, \ldots, r). \tag{4.6}
\]

Notice that the constraints on the controls come naturally from equation (2.8).

3. Every state interval \([T^\text{min}_i, T^\text{max}_i]\) and control interval \([c^\text{min}_k, c^\text{max}_k]\) is discretized into \(N_{T_i}\) and \(N_{c_k}\) intervals, respectively; the the spacing of the state component \(T_i\) is

\[
N_{T_i} = \frac{T^\text{max}_i - T^\text{min}_i}{\Delta T_i} + 1, \quad (i = 1, \ldots, n), \tag{4.7}
\]

and it is chosen in such a way that \(N_{T_i}\) is integer. The spacing of the control component \(c_k\) is obviously \(\Delta c_k = 1\), and therefore \(N_{c_k} = 2\).
Figure 4: The controlled evolution of the core temperature (solid line) obtained by applying the Dynamic Programming algorithm. The set point is \( T_b = 300 \text{ K} \), the performance index is \( J = 8.6 \text{ K}^2 \), and the daily average of the core temperature is 300.05 K. As a reference, the time evolution of the external temperature (dashed line) is also shown.

4. At every time step the Principle of Optimality is applied by proceeding backwards in time. This procedure gives the optimal control and optimal trajectory.

By memorizing a grid composed of \( M_{\text{grid}} \) points,

\[
M_{\text{grid}} = N \prod_{k=1}^{r} N_{c_k} \prod_{i=1}^{n} N_{T_i} = N \cdot 2^r \prod_{i=1}^{n} N_{T_i}, \tag{4.8}
\]

any possible control can be applied to any possible state, where "possible" is meant in the sense of equation (4.6).

The outcome of the application of the DP algorithm to the system (3.3) is reported in figure 4. By applying DP one may include in advance every possible perturbation from the optimal motion, and therefore one has closed-loop control. However, a digital computer with large memory capability is required due to the fact that we have a grid of \( M_{\text{grid}} \) stored data. In the present case \( M_{\text{grid}} \) can be shown to be huge. The "curse of dimensionality" of DP [1] is a strong motivation for the introduction of the Thermodynamic Automaton control in the next section.

5. The Thermodynamic Automaton

5.1 The Thermodynamic Automaton law

The ThA law is a feedback control, or in other words a strategy that gives the bang-bang controls as explicit functions of easily measurable quantities such as the state functions \( T_i \) and the driving term \( T_E \) (we exclude \( \phi_{\text{in}} \) from the arguments of \( C \) because otherwise that would imply the use of non-trivial
instrumentation). Therefore a ThA switching function must be of the kind $(ij = 12, 23, 3E)$

$$\varphi_{ij} = \varphi_{ij} [T, T_E; T_b].$$

(5.1)

The proposed switching function is

$$\varphi_{ij} = (T_1 - T_b)(T_i - T_j),$$

(5.2)

giving the control law that we call the Thermodynamic Automaton law:

$$c_{ij} = \Theta [(T_1 - T_b)(T_i - T_j)],$$

(5.3)

represented as a flow chart in figure 5. The ThA is therefore conceivable as a physical device that is able to handle any sort of a priori unknown arguments.

Physically, the meaning of the ThA law is the following. The first input to $c_{ij}$ is the error

$$e(t) = T_1(t) - T_b.$$

(5.4)

The second input is the sign of the heat flux $q^*_{ij}$ between compartments $i$ and $j$. From equation (2.4) we see that

$$\text{sgn} q^*_{ij} = \text{sgn} (T_i - T_j),$$

(5.5)

and therefore we arrive at the following very general ThA law encompassing equation (5.3):

$$c_{ij} = \Theta [e q^*_{ij}].$$

(5.6)

The ThA control (5.3) is a map of the kind

$$c_{ij} : (T_1, T_i, T_j) \rightarrow \{0, 1\}.$$
Figure 6: The controlled time evolution of the core temperature $T_1$ of four different controlled systems corresponding to $T_b = 290, 300, 310, 320 \, \text{K}$, respectively (continuous lines). Compare the uncontrollable time evolution of the external temperature $T_E$.

Figure 7: The time evolution of the compartmental temperatures $T_i$ (above) and the parallel time evolution of the control functions $c_{ij}$ (below), in correspondence to the set point $T_b = 300 \, \text{K}$.
Equations (5.3) and (5.7) imply that, by varying $T_b$, we generate a family of maps with parameter $T_b$ and correspondingly a family of controlled dynamical systems $Z(4, 3)$ with parameter $T_b$.

Figure 6 shows a set of curves $T_1(t)$ with parameter $T_b = 290 \div 320$ K. By comparing figure 6 with figure 4, we see that the performance of the systems controlled by the ThA law is identical to that of the system controlled by the DP algorithm. Finally, figure 7 shows the time evolution of temperatures and controls when $T_b = 300$ K.

5.2 Performance indices for the system $Z(4, 3)$

The performance of a controlled dynamical system corresponding to a certain set point $T_b$ can be condensed in a few quantities calculated over $\tau$ such as

- the performance index $J$,
- the mean daily excursion $\Delta T_1$, and
- the mean daily value $\langle T_1 \rangle$ of the core temperature.

By varying $T_b$ we obtain an ensemble of $Z(4, 3)$ systems and a corresponding set of curves of the kind $J = J(T_b)$, $\Delta T_1 = \Delta T_1(T_b)$, and $\langle T_1 \rangle = \langle T_1 \rangle(T_b)$. Notice that the driving terms $\{T_E, \phi^{in}\}$ are a set of functions $G$ that is defined once and for all; therefore an alternative approach would be to hold $T_b$ constant (e.g., $T_b = 304.16$ K) and to vary $G$, thereby simulating different seasonal and geographical conditions.

It is useful to use as a performance index the root mean square deviation of $T_1$ from $T_b$: $J = \sqrt{\mathcal{J}}$, which has the dimension of a temperature. Figure 8 gives the performance index $J(T_b)$.

Moving out of the range $280$ K $\leq T_b \leq 370$ K, many characteristic parameters of the system become insensitive to any further variation of $T_b$ although the controls are still acting. Two such parameters are $\langle T_1 \rangle = \langle T_1 \rangle(T_b)$, shown in figure 9, and $\Delta T_1 = \Delta T_1(T_b)$, shown in figure 10.

Another important parameter is the number $N^{sw}$ of times that the controls switch during a period $\tau$. In a discrete-time simulation, with a subdivision of $\tau$ into $N$ time intervals, $N^{sw}$ is defined by

$$N^{sw} \equiv \sum_{ij=12,23,3E} \sum_{p=1}^{N} |c_{ij}(p) - c_{ij}(p-1)|. \quad (5.8)$$

The phenomenon of chattering refers to a very large $N^{sw}$. Chattering is notoriously eliminated by introducing a floating interval with semi-amplitude $\eta$, which corresponds to the following modification of the error $e$:

$$e(t) = (T_1 - T_b) \longrightarrow e_\eta(t) = e \Theta(|e| - \eta). \quad (5.9)$$

Figure 11 shows $N^{sw} = N^{sw}(T_b, \eta)$. As intuitively expected, $N^{sw}$ diminishes with increasing $\eta$. The dependence on $T_b$ is also rather intuitive: the set point
Figure 8: The dependence on $T_b$ of the performance index $J$, the root mean square deviation of the core temperature from the set point temperature. Notice that $J(T_b)$ has its minimum at $T_b \approx \langle T_E \rangle$.

Figure 9: The daily average $\langle T_1 \rangle$ of $T_1$ as a function of the set point $T_b$. Notice that the curve $\langle T_1 \rangle$ obtained numerically stays close to the theoretical optimum curve defined by $\langle T_1 \rangle \equiv T_b$ in the range $285 \leq \langle T_1 \rangle \leq 330$ K. The plateaux at the right of 370 K and at the left of 280 K $\approx \min T_E$ correspond to the saturation of the control activity.
Figure 10: The daily excursion $\Delta T_1$ of the controlled core temperature $T_1$, as a function of the set point temperature $T_b$. The minimum of $\Delta T_1$ is at $T_b \approx 300\,\text{K} \approx \langle T_E \rangle$. Similarly to figure 9, the plateaux at the right of 370 K and at the left of 280 K $\approx \min T_E$ correspond to a saturation in the control action.

Figure 11: The dependence of the number of control switchings per period $N_{\text{sw}}$ on the floating interval amplitude $\eta$ and the set point $T_b$. 
Figure 12: The dependence of the performance index $J$ on the floating interval amplitude $\eta$ and the set point $T_b$.

intervals in which $\mathcal{N}_{\text{sw}} \geq 0$ correspond to the above-mentioned insensitivity regions of $\Delta T_1$ or $\langle T_1 \rangle$.

The presence of the floating interval is essential for a realistic simulation of the control system. In fact the error $e$ in equation (5.4) corresponds to an assumption of infinite precision in the measurement of the compartmental temperature $T_1$, which in turn would underlie infinitely rapid equilibration of the temperature field inside compartment 1. Therefore the error $e_\eta$ takes into account the inevitable delays between control and system dynamics. For instance, figure 7 was obtained with $\eta = 2$ K. Figure 12 shows $J = J(T_b, \eta)$, whose projection onto the plane defined by $\eta = 0$ is shown in figure 8.

5.3 Thermodynamic Automata and Cellular Automata

The relationship between Cellular Automata (CAs) and the Thermodynamic Automata (ThAs) discussed in this paper is the following.

1. A ThA is a finite set of subsystems. The interactions are heat flows. These flows are modulated by bilinear controls that map the state space onto the $r$-fold cartesian product $[0, 1] \times [0, 1] \times \cdots \times [0, 1]$.

2. A CA is an array of interacting cells. The interaction determines the system evolution. The interaction may be suggested by different evolution schemes: differential reversible or irreversible (dissipative), deterministic or probabilistic. The interaction is assigned by means of memorized tables.

The analogy between ThAs and CAs is bound to a reduction of the system into a finite set of elements: the compartments in the case of Thermodynamic Automata, the cells in the case of Cellular Automata. However, the compartments of the thermodynamic model are complex entities themselves
characterized by a continuous set of states, whereas the cells are elementary entities that may live on \( k \) different states.

There is a stronger analogy between ThA controls and CA evolution. Both are regulated by decisional tables (flow charts).

In conclusion, the analog of the ThA subsystem is the CA cell. The analog of the ThA control tables are the CA evolution tables.

It is interesting to study the evolution of the controlled dynamical system in the state space. First of all it is to be expected that all solutions are attracted to a regular periodic orbit both in controlled and uncontrolled dynamics [2]. Here we limit ourselves to consider the periodic steady-state behavior, having numerically exhausted all transients. The problem is how to visualize the motion \( T(t) \). It is useful to study the projection of the trajectory \( T(t) \) onto the three spaces of the kind \( (T_i, T_j) \) corresponding to the "coupling" of states actuated by each control. As an example we show in figure 13 the projection of \( T \) on the space \( (T_1, T_2, T_3) \) with \( T_b = 300 \) K, and (a) \( \eta = 0 \) K or (b) \( \eta = 2 \) K. The trajectory is further projected onto the planes \( (T_1, T_2) \) and \( (T_2, T_3) \).

6. Entropy balance of \( Z(4,3) \)

The scope of the present section is to understand analytically and numerically the influence of the controls on the entropy production and flux of the system.

If \( U_i(t) \) denotes the internal energy of the \( i \)th compartment, the entropy rate for the \( i \)th compartment is \( \dot{S}_i = \dot{U}_i/T_i = \mu_i T_i/T_i \) and the total entropy rate is \( \dot{S} = \sum_i \dot{S}_i \). The total entropy balance (in W/K) is

\[
\dot{S}(t) = \Sigma(t) + \phi_{\text{in}}^S(t) - \phi_{\text{out}}^S(t),
\]
where $\Sigma$ is the total entropy production and $\phi^{in}_B - \phi^{out}_B$ the net entropy flux. From equations (3.2) and (3.3) we deduce

$$
\dot{S}(t) = [\beta_{12} + \tilde{\beta}_{12} c_{12}(t)] \frac{[T_1(t) - T_2(t)]^2}{T_1(t) T_2(t)}
+ [\beta_{23} + \tilde{\beta}_{23} c_{23}(t)] \frac{[T_2(t) - T_3(t)]^2}{T_2(t) T_3(t)}
+ [\beta_{3E} + \tilde{\beta}_{3E} c_{3E}(t)] \frac{[T_E(t) - T_3(t)]}{T_3(t)}
+ \beta_{1E} \frac{[T_E(t) - T_1(t)]}{T_1(t)}
+ S_{4E} \frac{\phi^{in}_E(t)}{T_4(t)} - S_{4E} \sigma T_4^3(t).
$$

The total entropy production is given by the first three terms and the net entropy flux by the remaining terms. The entropy manipulation performed by the control is evident in the above equations:

1. The entropy production is explicitly affected by the inner controls (figure 2) $c_{12}$ and $c_{23}$ only.
2. The entropy flux is instead affected by the boundary control $c_{3E}$ only.

In figure 14 the daily average entropy production $\langle \Sigma \rangle = \langle \Sigma \rangle (T_b)$ is shown. We see also that $\langle \Sigma \rangle$, like $\langle T_1 \rangle$ and $\Delta T_1$ (figures 9 and 10), has a range of sensitivity on the set point.
As a final remark, notice that $S(t)$ is periodic due to the periodicity of the driving terms $T_E$ and $\phi^{\text{in}}$. Therefore we deduce from equation (6.1) that
\[ \langle \dot{S} \rangle = 0 = \langle \Sigma \rangle + \langle \phi^{\text{in}}_S \rangle - \langle \phi^{\text{out}}_S \rangle. \] (6.3)

Extensive numerical simulations have shown that
\[ \langle \Sigma \rangle \ll \langle \phi^{\text{in}}_S \rangle, \langle \phi^{\text{out}}_S \rangle, \] (6.4)
and therefore equation (6.3) is satisfied mainly due to the balancing of $\phi^{\text{in}}_S$ and $\phi^{\text{out}}_S$.

7. Concluding remarks

In this theoretical analysis supported by numerical calculations, we have shown how $Z(4,3)$ satisfies the imposed variational requirement. Therefore the goals stated in the introduction are meaningful and the related problems are, at least in principle, solvable.

It is clear that this work does not yet give a solution to realistic problems because we adopted drastic simplifications. On the other hand, this model might act as a stimulation for studying more complex dynamical systems.

In particular, what we learn from the study of $Z(4,3)$ is the following. The performance index (2.1) could be reduced, as intuitively expected, by increasing the structural mass $m_2$ and by decreasing the heat transmission coefficient $\alpha_{1E}$ pertinent to the heat exchange between 1 and E. Figure 15 shows to this purpose the root mean square deviation $J = J(\alpha_{1E}, m_2)$ with $T_b = 300$ K and $\eta = 0$ K.

However, the phenomenon of the "bumps" and "dips" of $T_1(t)$ with respect to $T_b$ (see figure 6) tends to persist, as repeated simulations have shown.
This is an intrinsic defect of the feedback approach: the system optimizes over the instantaneous behavior and wastes the internal energy reserves in the intermediate ranges of $|T_1(t) - T_b|$; thus it remains unguarded against the stronger oscillations. In other words, the ThA is improvident.

More complex compartmental reductions of the kind $Z(n, r)$—where $n$ is the number of compartments and $r$ the number of controls—can be considered with their resulting control flow charts. In this way one can develop a control thermodynamics with more sophistication and also a better adherence to realistic problems.

The next theoretical step is to introduce adaptive controls with a smooth and a more slowly varying dynamics than the "improvident" feedback controls; that is, adaptive thermodynamical structures of the kind $Z(n, r, x)$, where $x$ represents the adaptive controls. Nevertheless, we think that the present study of $Z(4,3)$ contains the spirit of the general approach, while remaining at the same time almost "intuitive."

Appendix A. The driving terms

The input functions of the problem are the external air temperature $T_E$ and the incoming radiation flux $\phi_{\text{in}}$ on the receiving surface (compartment 4).

In addition to being chaotically time dependent, these data also show an amount of periodicity. We seek to simulate the regular periodic part. This appendix is conceptually necessary only for the application of the open-loop control à la Pontrygin, given by equation (4.4).

In fact, the Thermodynamic Automata operation (section 5) as well as Dynamic Programming (section 4) are feedback controls, and therefore analytic expressions for $T_E$ and $\phi_{\text{in}}$ are not necessary. It is, however, convenient to use the following model inputs also for a preliminary theoretical study of the Thermodynamic Automaton.

Radiation

The flux of impinging solar radiation (in W m$^{-2}$) is simulated by

$$\phi_{\text{in}}(t) = g(t) + \sigma T_E^d(t), \quad (A.1)$$

where $\sigma$ is the Stefan-Boltzmann constant. In this equation, $g$ represents the flux of direct radiation with time-dependent geometry; $\sigma T_E^d$ represents the geometry-independent diffuse flux. The direct component $g$ is in first approximation described by an analytic function (see figure 3).

The parameters characterizing $\phi_{\text{in}}$ are $\langle g \rangle \simeq 375$ W m$^{-2}$, $g = \max g \simeq 630$ W m$^{-2}$ at $t = 12^h00^{\text{min}}$; $g > 0$ only when $4^h00^{\text{min}} < t < 20^h00^{\text{min}}$.

Temperature

The connection between $T_E$ and $g$ is in general extremely complicated since it implies knowledge of the global atmospheric dynamics of the Earth. The
very first approximation is to simulate $T_E$ with a sinusoid. The second approximation is to consider $T_E$ as related deterministically to $g$ by means of a differential equation that represents essentially the instantaneous energy balance for a flat horizontal surface (in W):

$$\mu_E T_E = S_E [g(t - t_{lag}) - \varepsilon_E \sigma T_E^4(t)].$$ \hspace{1cm} (A.2)

Notice that only radiation energy exchanges are taken into account. In equation (A.1) the coefficient $\varepsilon$ is $0 < \varepsilon_E \leq 1$, and $\mu_E$ is heat capacity in J/K. How do we assign $\varepsilon_E$ and $\mu_E$? We ask equation (A.2) to give solutions simulating a representative day with a given the mean daily value of the ambient temperature $\langle T_E \rangle$ and the daily oscillation $\Delta T_E$. To obtain the given data, we manipulate $\varepsilon_E$ and $\mu_E$, for instance

$$\varepsilon_E = 0.57, \quad \mu_E = 3.15 \times 10^5 \text{ J/K} \implies \langle T_E \rangle = 298.0 \text{ K}, \quad \Delta T_E = 30.2 \text{ K}.$$ \hspace{1cm} (A.3)

The value $\langle T_E \rangle$ is suggested by experimental data at latitude 45° and is consistent with the choice for $g$ whereas $\Delta T_E$ is very large. Notice that $t_{lag}$ appearing in equation (A.3) affects only the distance between max $T_E$ and max $g$, but not the average values. The outcome is shown in figure 3.

**Appendix B. Parameters**

**Heat capacity**

The heat capacity of compartment $i$ is given by $\mu_i = c_{v,i} m_i$ where $m_i$ is mass and $c_{v,i}$ is specific heat of compartment $i$:

$$\mu_1 = 1300 \text{ J/K}, \quad \mu_2 = 46000 \text{ J/K}, \quad \mu_3 = 156 \text{ J/K}, \quad \mu_4 = 460 \text{ J/K}.$$ \hspace{1cm} (B.1)

The influence of the heat capacity $\mu_2$ on the performance of the system is fundamental. Since no additive controls are present, the heat sinks and sources of the core are the external ambient (uncontrolled flow $q_{1E}$) and the structural compartment 2 (controlled flow $q_{12}^r$). The larger $\mu_2$, the smaller the oscillations $\Delta T_1$; but also the transient will be longer. Figure 16 shows the evolution of $T_1(t)$ with $T_b = 300 \text{ K}$ over $2 \tau$ in correspondence to four values of $\mu_2$, starting from the initial condition $T_i(t_0) = T_E(t_0)$ ($i = 1, \ldots, 4$) where $t_0$ is defined to be midnight.

**Overall heat transmission coefficient**

In our model shown in figure 2 the compartments 1 and 3 are necessarily fluid (air), while 2 and 4 are solid (iron and copper, respectively). If a pair of compartments $(i, j)$ is separated by a homogeneous solid layer with thickness $\ell_{ij}$ and thermal conductivity $\kappa_{ij}$, the solution of the one-dimensional Fourier equation gives notoriously the conductive heat flow (in W)

$$q_{ij} = \frac{\ell_{ij}}{\kappa_{ij}} S_{ij} (T_i - T_j).$$ \hspace{1cm} (B.2)
The overall heat transmission coefficient $\alpha_{ij}$ relative to the $(i, j)$ interface is defined by

$$\frac{1}{\alpha_{ij}} = \frac{1}{h_i} + \frac{\ell_{ij}}{\kappa_{ij}} + \frac{1}{h_j}$$

in such a way that we can express the conductive-convective heat flow between $i$ and $j$ as in equation (2.2), in formal analogy to equation (B.2). The phenomenological linear expression (2.2) contains both Newton’s “cooling” law ($h$ coefficients) and the exact solution of the one-dimensional Fourier equation for a homogeneous wall ($\kappa$ coefficients). The $\beta_{ij}$ and $\tilde{\beta}_{ij}$ coefficients defined in equation (2.4) are

$$\left\{ \begin{array}{ll}
\tilde{\beta}_{12} = 25.0, & \beta_{34} = 50.0, \quad \tilde{\beta}_{3E} = 60.0 \text{ W/K (mainly convection)}; \\
\beta_{3E} = 2.0, & \beta_{12} = \beta_{23} = 0.2, \quad \beta_{1E} = 0.8 \text{ W/K (mainly conduction)}.
\end{array} \right. $$

Influence of $\tilde{\beta}_{ij}$ on the control performance

The equilibration between $T_i$ and $T_j$ as $c_{ij} = 1$, and therefore the performance of the control action, is expected to depend on the ratios $\tilde{\beta}_{ij}$—more precisely on $\tilde{\beta}_{ij}/\mu_i$ and $\tilde{\beta}_{ij}/\mu_j$ (this is intuitive from equation (2.4) and the beginning
Figure 17: The dependence of the performance index $J$ of two controlled systems (corresponding to the set points $T_b = 295$ K and $T_b = 300$ K) on the parameter $\tilde{\beta}_{ij}$ ($ij = 12, 23$), which is the heat transmission coefficient pertinent to the control $c_{ij}$. The parameter $\tilde{\beta}_{3E}$ has been set arbitrarily to $2.4 \times \beta_{12}$.

of section 3). The coefficient $\tilde{\beta}_{ij}$ is very difficult to define theoretically and in an experimental situation would depend on many factors, such as the relative disposition of the compartments (free convection) and the presence, say, of small electric fans that would hugely enhance the $\tilde{\beta}_{ij}$ (forced convection).

Figure 17 shows the dependence of the performance index $J$ on $\tilde{\beta}_{ij}$ ($ij = 12, 23$) in correspondence to two set points near $\langle T_E \rangle$. We held the ratio $\tilde{\beta}_{3E}/\tilde{\beta}_{ij}$ arbitrarily fixed to 2.4 and the ratio $\beta_{34}/\tilde{\beta}_{ij}$ fixed to 2 according to figure 2.

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