Behavior of Topological Cellular Automata

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Abstract. We introduce a new type of cellular automaton, one in which the link structure is dynamically coupled to the site values. The automaton structures are altered using simple Boolean rules, while the sites themselves are assigned values based on the mod 2 rule. We compare these dynamics to those in which the link structure is altered randomly and find that in the former case stable structures of noninteger dimensionality emerge. Fully exploring this model, we observe the effects of value rule alteration, initial lattice structure alteration, and alteration in the initial value seeding and observe patterns of self-organization, growth, decay, and periodicity. Finally, we comment on the relationship between this model and randomly generated Kauffman nets.

1. Introduction

In recent years it has been shown that cellular automata provide means of modelling a wide range of physical systems [1]. Typically, automaton dynamics are determined by an initial value seeding (with site values specified from a certain range) and a set of simple, local transition rules. This behavior takes place within a fixed lattice structure (e.g., each site is linked to two, four, six, or more neighbors).

Ilachinski has pointed out the limitations of these *structurally static* automata and has suggested a scheme for *structurally dynamic* models [2]. Here, the lattice structures are dynamically coupled to the local site value configurations. A preliminary study of some of these models was completed by Ilachinski and Halpern, yielding evidence of a wide range of behavior [3].

In this paper we wish to explore a new sort of *topological automaton* (TA) model, one in which lattice dynamics are determined by simple Boolean rules. In this approach, both the site values and the underlying lattice structures are treated in a similar manner, creating a nonlinear "feedback" mechanism determining future automata states. Thus, one can use topological automata to search for geometric self-organization.

Although we wish to present a purely mathematical model, possible physical applications for TAs are numerous. These include schemes for crystal growth, pattern formation, and types of neural network models. While one can readily model crystal growth using conventional "value" cellular automata, topological automata can more precisely define an amorphous crystal structure determined by local interactions.

In addition, lattice gas models might be constructed by use of topological automata. In particular, schemes might be considered in which the value structure and link structure of a topological automaton represent two different interacting substances [4].

Topological automata may also provide a means of describing the "geometric exciton" program of Wheeler [5], in which all particles are considered to be geometric disturbances, with space itself seen as emerging from a "pregeometry" of indeterminate dimensionality. Since "value structure" solitons have been found in conventional cellular automaton models [6], "link structure" solitons might emerge in a TA scheme for geometrodynamics. TA models could then describe the generation, transmission, and interaction of topological disturbances.

Finally, there appears to be some relationship between TAs and Kauffman nets used in evolutionary genetic theory [7]. We shall briefly comment on this latter application in our conclusion.

Let us now formally define a Boolean topological automaton. Consider a network of N sites. Each of these sites may have value 0 or 1. Therefore at a given time t we define the value structure of an automaton by the matrix v_i^t ($i = 1, N$). We further define the link structure by the connectivity matrix l_{ij}^t ($i = 1, N; j = 1, N$). Then two sites i, j can be described as linked if and only if $l_{ij}^t = 1$ (otherwise $l_{ij}^t = 0$). In this case the sites can be said to be neighbors.

We may also define the next-nearest neighbors of each site by use of the secondary connectivity matrix m_{ij}^t. Two sites i, j are defined as next-nearest neighbors if and only $m_{ij}^t = 1$ (otherwise $m_{ij}^t = 0$). The matrix m_{ij}^t is completely determined by l_{ij}^t in the following manner:

$$m_{ij}^t = 1 \leftrightarrow \exists k : l_{ik}^t l_{kj}^t [1 - l_{ij}^t] = 1 \tag{1.1}$$

Now we can define a set of transition rules, which can be grouped into value transition rules and link transition rules (couplers and decouplers). There are two sorts of value rules that we wish to consider, totalistic and outer-totalistic.

Totalistic value rules may be defined in the following manner:

$$
\begin{aligned}
\text{Given a set} \quad & [\alpha], \text{ then: } v_i^{t+1} = \phi_{[\alpha]}\left(\sum_{j=0}^N l_{ij}^t v_j^t, v_i^t\right) \\
\text{where} \quad & \phi_{[\alpha]}(x, a) = \sum_\alpha \delta(x + a, \alpha) \\
\text{and} \quad & \delta(x, y) \text{ is the kronecker delta.}
\end{aligned}
\tag{1.2}
$$

Outer-totalistic value rules may be defined as follows: In a similar manner we wish to define a *decoupler*, which removes a link, and a *coupler*, which adds a link. The set of all

$$\text{Given sets} \quad [\alpha_0], [\alpha_1], \text{ then: } v_i^{t+1} = \phi_{[\alpha_0],[\alpha_1]} \left(\sum_{j=0}^{N} l_{ij}^t v_j^t, v_i^t \right)$$
$$\text{where} \quad \phi_{[\alpha_0],[\alpha_1]}(x, a) = a \sum_{\alpha_1} \delta(x, a_1) + (1 - a) \sum_{\alpha_0} \delta(x, \alpha_0)$$
(1.3)

couplers and decouplers determines our link transition rules.

Boolean decouplers are defined as follows:

$$\text{Given} \quad l_{ij}^t = 1, \text{ then } l_{ij}^{t+1} = \Phi(l_{ij}^t),$$
$$\text{where} \quad \Phi = 1 - \delta(v_i^t + v_j^t, 0)$$
(1.4)

Thus, two linked sites i and j are decoupled if and only if the sum of their site values is zero.

Boolean couplers are defined in the following manner:

$$\text{Given} \quad l_{ij}^t = 0 \wedge m_{ij}^t = 1,$$
$$\text{then} \quad l_{ij}^{t+1} = \omega(l_{ij}^t)$$
$$\text{where} \quad \omega = \delta(v_i^t + v_j^t, 2)$$
(1.5)

so, two unlinked sites i and j become linked if and only if they are next-nearest neighbors and the sum of their site values is two.

Therefore, a Boolean topological automaton is fully defined by an initial state vector,

$$|G\rangle_0 = |v_1^0, \ldots, v_N^0; \{l_{ij}^0\}\rangle$$
(1.6)

and a set of transition rules linking sequential state vectors:

$$|G\rangle_{t+1} = \prod_i \Phi_{[\alpha]} \prod_{(l_{ij}^t = 1)} \Psi \prod_{(m_{ij}^t = 1)} \omega |G\rangle_t$$
(1.7)

representing the operator product of all value transitions, decouplers, and couplers applied to the state vector at time t. Note that all transitions occur simultaneously.

These link rules have been chosen for their simplicity and applicability. It is natural to think of systems in which two disconnected but close active cells (sites of value 1) form a bond and two inactive cells (sites of value 0) lose their connection (one might keep in mind certain types of molecular bonding, for instance). We shall comment further on this choice of link rules in the conclusion.

Let us now consider a simple example of the application of these rules. Let us assume that our initial lattice state is a two-dimensional 3×3 lattice in which each site (aside from the bordering ones) has four neighbors. We populate this initial state with values of 0 and 1 in the following manner (see figure 1):
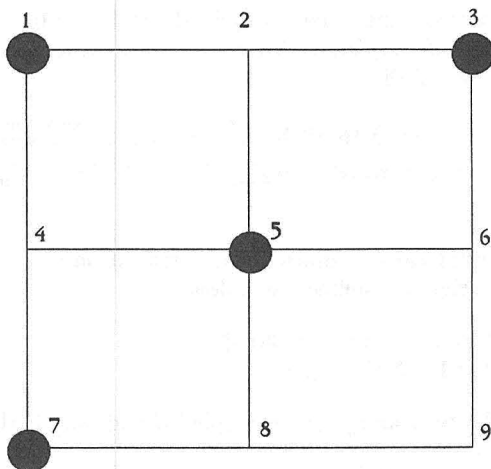
Figure 1: A 3×3 example of a Boolean topological automaton. This represents the initial configuration of the lattice, with the darkened circles representing ones and the other points representing zeroes.

$$
\begin{aligned}
v_1^0 &= v_3^0 = v_5^0 = v_7^0 = 1 \\
v_2^0 &= v_4^0 = v_6^0 = v_8^0 = v_9^0 = 0
\end{aligned} \tag{1.8}
$$

Next, we apply the totalistic "mod 2" value rule, our decoupler rule, and our coupler rule to this initial state. The mod 2 value rule assigns a new value to a lattice site based upon a neighborhood sum of the old values (0 if the sum is even, 1 if odd). The decoupler rule removes all links connected to sites in which both values are zero. The coupler rule adds links between all next-nearest-neighbor sites in which both values are one. All of these actions occur at the same time.

Therefore if we look at the lattice after the first time step (see figure 2) and compare it to its previous state, we find that the site value configuration has been altered. Since, before the first time step, the neighborhood sums of sites 2 and 4 are odd, the values of these sites become 1.

In addition, the decoupler acts to remove the link between sites 6 and 9, since the values at the endpoints of this link are both zero. For the same reason, the link between sites 8 and 9 is removed during time step 1.

Finally, let us consider the actions of the coupler. Five links are added during the first time step. Links are added between sites 1 and 3, 1 and 4, 1 and 7, 3 and 5, and 5 and 7, since in each of these cases a set of next-nearest neighbors has value 1 for both sites.

We should note that in this example and in our studies we have assumed that the operators ω, Ψ, and Φ are applied simultaneously to the automaton state $|G\rangle$. Alternatively, one might imagine a time-ordering of these operators, in which the value rule would be applied first, followed by the link rules. Clearly, this would alter the automaton dynamics.
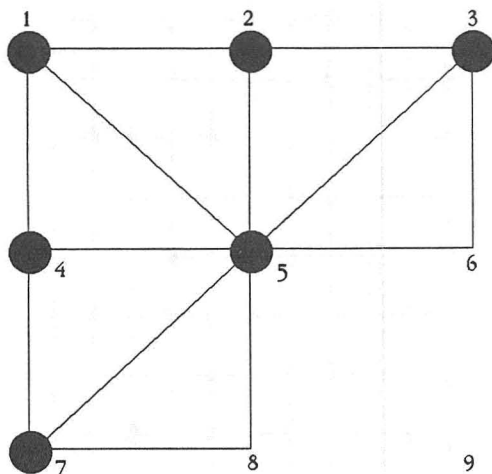
Figure 2: The example, following the first time step. Note that two links have been deleted by decouplers and five links (two not seen) have been added by couplers.

2. The generation of geometric patterns

We now wish to consider the patterns generated by the application of the automaton rules to a two-dimensional Euclidean lattice. We consider both randomly binary seeded lattices and lattices in which the initial value structure is represented by a 3×3 block of site value one surrounded by a "sea" of zeroes. We assume that each site starts out with four neighbors and further assume periodic boundary conditions in which the lattice topology starts out as a 2-torus. For clarity in presentation, we look at lattices of 121 sites (11×11). Our results in this section can readily be generalized to larger lattices.

In our pictorial representations of the automaton dynamics, we depict the links as line segments connecting the sites. Note, however, that these links may overlap, in which case it may be difficult to discern which sites are connected. Fortunately, though, this does not alter the qualitative picture that emerges.

We have found several distinct types of patterns, depending upon the value rule used and the initial seeding of the lattice. In many cases, when the lattice was seeded randomly, we found unrestricted growth in the number of links per site, until all sites were linked. In other cases, we found "clumping" and decay of the lattice to a simple state. We found examples in which the final state was stable and others in which it was simply periodic, i.e.:

$$|G\rangle = |G\rangle_{t+p} = \left[\prod_i \Phi_{[\alpha]} \prod_{(m_{ij}^t=1)} \omega \right]^p |G\rangle_t \qquad (2.1)$$
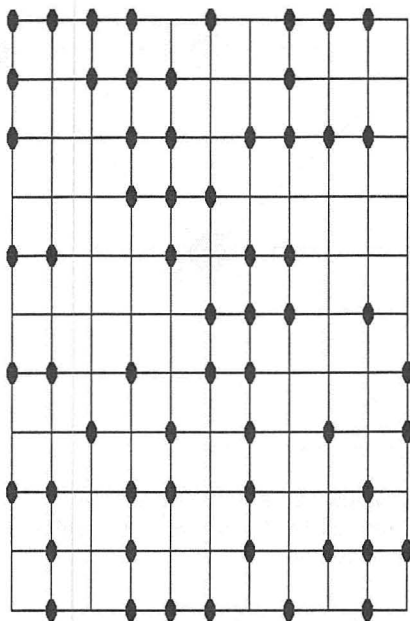
Figure 3: A graphical depiction of the initial state of a randomly seeded Cartesian lattice with periodic boundary conditions. Sites of value one are indicated by darkened circles.

$$p = 1, 2, 4, \text{etc.} \tag{2.2}$$

In figures 3–5 we see an example of the application of the mod 2 value rule to a randomly seeded lattice. We can see in this figure how after time step 1 a fairly large number of sites are linked and several are decoupled. By time step 5 there are an enormous number of links between sites, too numerous to observe in a distinct manner.

In figures 6–11 we see what happens when the mod 2 value rule is applied to a lattice seeded with a 3 × 3 block of ones surrounded by zeroes. Note that as the lattice structure evolves, it passes through stages in which it passes through a number of distinct patterns (one can bear in mind crystal formation). Finally, after time step 5, the automaton reaches a periodic state with a periodicity of 2 in the value and lattice structure.

In the last set of geometric images, figures 12–15, we see how the automaton behavior changes when the value rule is altered. Here a step function value rule is used:

$$v_i^{t+1} = 0 \quad \leftrightarrow \quad \sum_{j=1}^{n} l_{ij}^t v_i^t > 2 \tag{2.3}$$

$$v_i^{t+1} = 1 \quad \leftrightarrow \quad \sum_{j=1}^{n} l_{ij}^t v_i^t \leq 2 \tag{2.4}$$
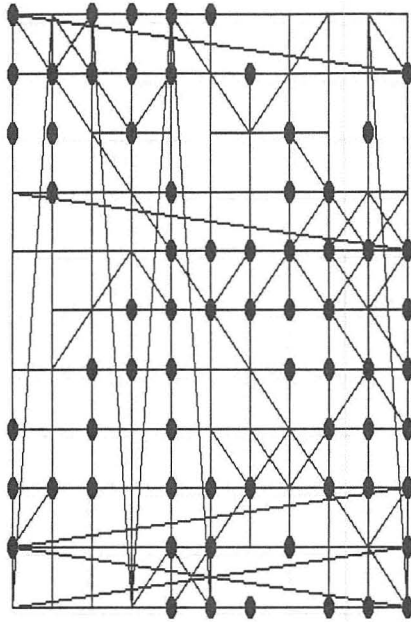
Figure 4: The link rules and the mod 2 value rules have now been applied. Note that the number of links per site has increased.
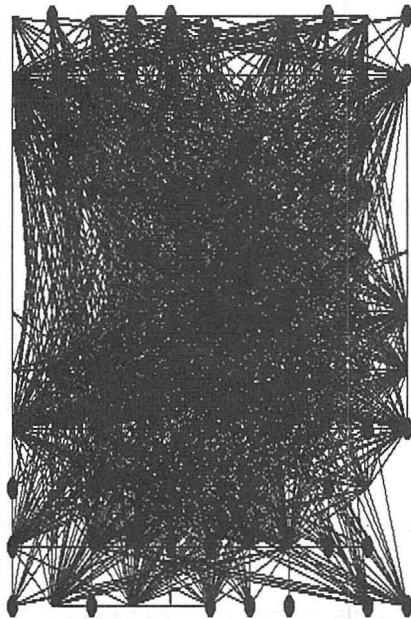

Figure 5: By time step 5, the randomly seeded lattice has so many connections that it is difficult to discern which sites are connected.
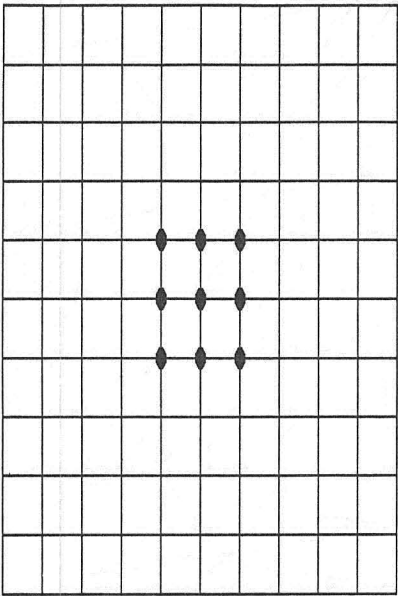
Figure 6: The value and link rules here are the same as in figure 3. However, the initial value configuration is now a 3 × 3 block of ones.
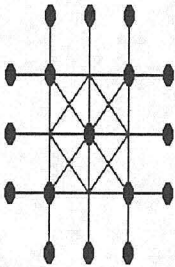

Figure 7: After the first time step, much of the lattice structure has decayed.

In contrast to the previous set of figures, we find that "clumping" takes place, with the values of the lattice frozen into small regions that are out of communication. The final state, after time step 4, is stable in both lattice and value configurations. Thus, we can see from figures 3–15 that there exists a dependence of pattern formation on both the value rule and the initial value seeding.
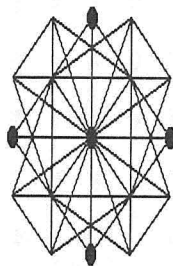
Figure 8: After the second time step there are few link and value changes.
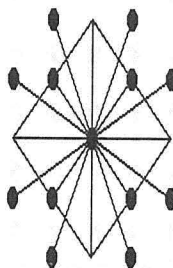
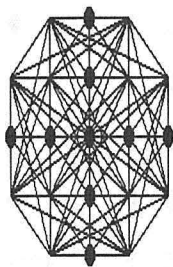Figure 9: Time step 3 for the mod 2 rule.
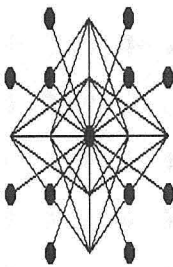

Figure 10: Time step 4 for the mod 2 rule.

Figure 11: By the fifth time step, the link and value structures have become periodic, with a periodicity of two.
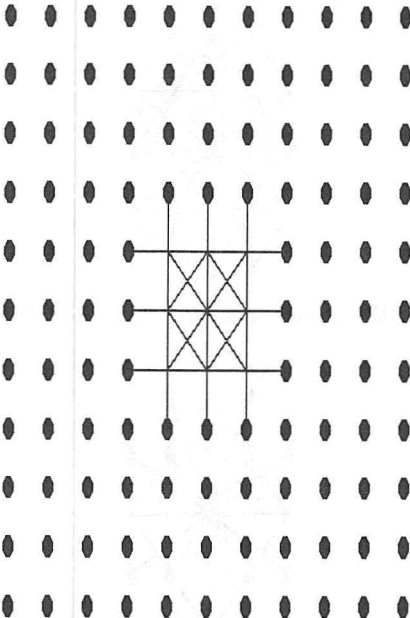
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Figure 12: Here is the step function value rule, applied to a 3×3 block of ones. Note that most of the links have been removed, but almost all the sites are "frozen" into values of one.
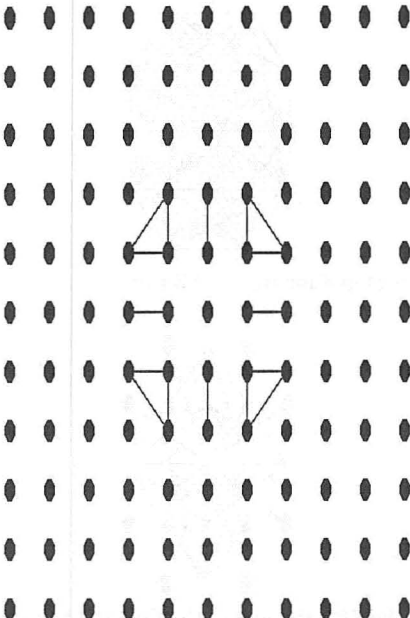

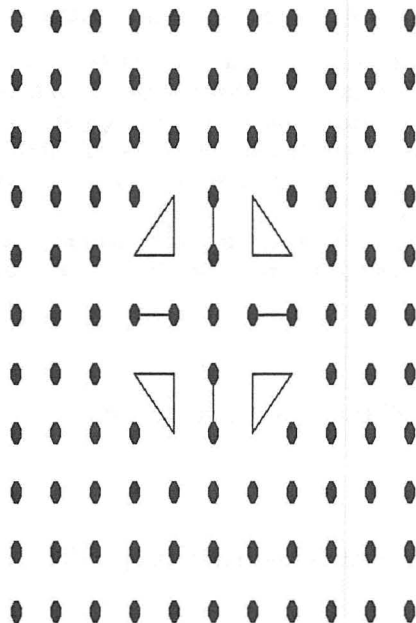Figure 13: After time step 2, there are few link or value changes.

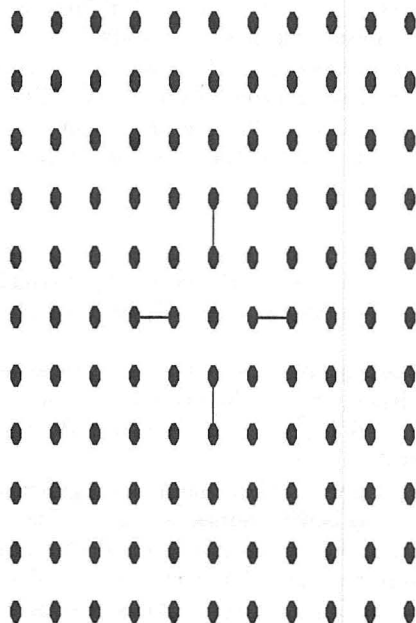Figure 14: After time step 3, the pattern is almost quiescent.

Figure 15: By time step 4, the automaton has reached a stable state.

3.   Statistical behavior of topological automata

We shall now examine the long-term behavior of Boolean topological automata as defined in the first section (equations 1.2–1.7). We have conducted an extensive survey of a large number of evolving systems (using a Ridge-32 computer) in which we have selected value rules, initial value configurations, and initial lattice structures and have made observations over a series of 100–10,000 time steps.

We have chosen a set of statistical measures in order to adequately explore this behavior. These include the value density (average site value)

$$\rho = \frac{1}{N} \sum_{i=1}^{N} v_i \tag{3.1}$$

the average number of links per site (average number of neighbors per site)

$$\gamma = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{ij} \tag{3.2}$$

the average number of next-nearest neighbors per site

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \tag{3.3}$$

and the average link length, computed by embedding the lattice in the two-dimensional plane and measuring the average size of each link. In addition, we have computed the total number of links deleted at each time step, the total number of links added, and the total number of link changes.

Finally, we have found the "effective dimension" at each time step, a discrete analogue of the continuous Hausdorff dimension:

$$D = \frac{\mu}{\gamma} \tag{3.4}$$

It may be shown [8] that this quantity, especially convenient for discussion of network properties, is identical to the Hausdorff-dimension in the finite link-length limit.

We have used these dynamic measures to investigate the evolution of networks of 15×15 sites configured in a two-dimensional square lattice with periodic boundary conditions. We then compared the results of these measures for different initial set-ups.

Our first configuration was a Cartesian four-neighbor lattice, seeded randomly (in all cases seeding is with zeroes and ones). The value rule we chose was the mod 2 mapping considered earlier. We then examined the effects of the link rules, by comparing evolution with the link rules to a complete updating of all links (adding links between all next-nearest neighbors, deleting links between all neighbors). In other words, in the "link rule" case, deletion and addition of links is value dependent, following equations 1.4 and 1.5. In

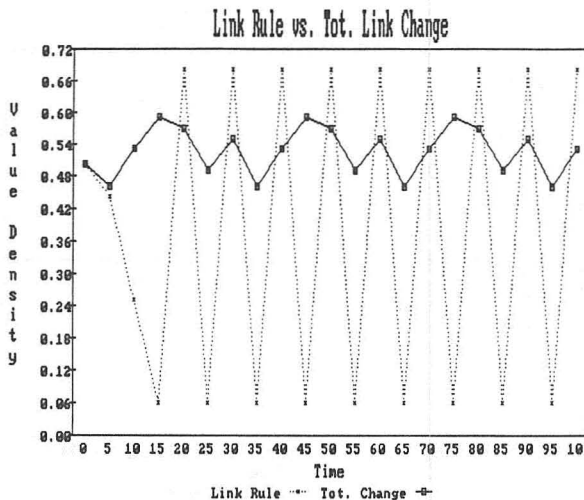**Link Rule vs. Tot. Link Change**

Figure 16: Here is a comparison of a topological automaton with Boolean, value-dependent link rules, to that of a system in which links are automatically added between all next-nearest neighbors and deleted between all neighbors after each time step. The value rule here is mod 2. Note that in the former case the value density has a period of two, whereas in the latter case the periodicity is 30.

the "complete updating case," all possible changes are made and there are no link rules. Thus, the latter serves as a control group to examine whether or not link rules have a significant effect.

The results of this comparison were quite interesting. Clearly, the link rules do have an effect: in the complete updating control group, the value density becomes periodic in later time steps, with a periodicity of 30, but if the link rules are considered, the value density reaches a periodicity of 2. In both cases, the links per site becomes periodic with a periodicity of 2, but for different reasons: in the complete updating case the decouplers and couplers serve to undo each other's actions after every two time steps, but in the link rule case, the periodicity is driven by a periodic value configuration. Finally, we should comment that the complete updating case evolves into a stable effective dimensionality of 1, whereas the link rule case never reaches a stable dimensionality. In figures 16 and 17, we see a comparison between the two cases.

Next we examined the effects of changing the value rule. We compared three different value rules and observed the influence of the link rule on the evolving networks. In the first case we used the mod 2 value rule on a four-neighbor Cartesian lattice. In the second case we looked at the step function value rule (equation 4). In the third case we applied Conway's life rule [9] to an eight-neighbor Cartesian lattice.
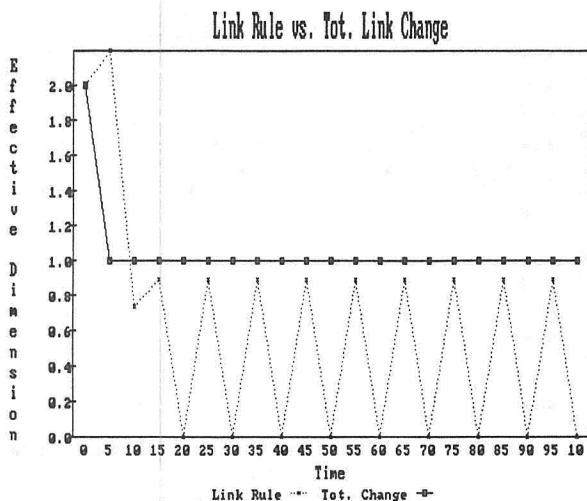
Figure 17: The "effective dimension," which is the number of next-nearest neighbors divided by the number of neighbors, reaches a stable state only when all possible link changes are performed, but not in the case where there are restrictive link rules.

In terms of value density, the behavior for each of these value rules is quite different. The mod 2 rule leads to long-term periodicity, oscillating between value densities of 0.68 and 0.06. On the other hand, the step function rule leads to long-term stability, with a final value density of one. Finally, the value density in the life rule reaches a stable value of 0.01. These are three qualitatively distinct sorts of evolution.

The link structure also evolves differently for the three rules. The mod 2 rule leads to a periodicity in lattice structure, oscillating between 103 and 12 links per site. The effective dimension is also periodic, oscillating between 0.89 and 0.01. However, for both the life and step function rules, there is a steady decay to a final state, which is almost completely disconnected (with average link length close to zero) and having a near-zero effective dimensionality. We can examine these behaviors in figures 18–20.

We also explored the effects of altering the initial lattice structure from a four-neighbor per site Cartesian pattern (or eight in the case of life) to a randomly connected network in which each pair of sites has a 50% chance of being connected in the initial state. For this comparison, we assumed a random initial site value configuration and a mod 2 value rule.

Our results can be seen in figures 21 and 22. Though starting out with a large number of connections (about 112 per site), a randomly connected lattice rapidly decays to the null lattice (no links, all values zero) within 15 time steps, in contrast to the Cartesian lattice, which as we have seen reaches periodicity.
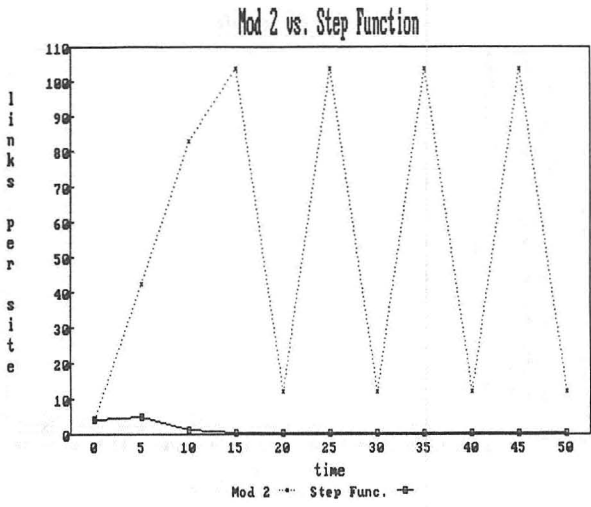
Mod 2 vs. Step Function

Figure 18: This graph shows how the choice of value rule strongly influences the long-term behavior of these systems. Here, the link structure "dies out" for the step function rule, but is periodic for the mod 2 rule.
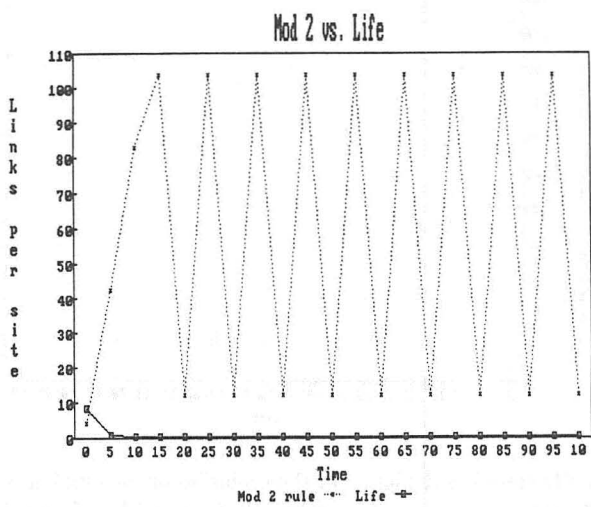
Mod 2 vs. Life

Figure 19: Here is another comparison, this time between Conway's life rule and mod 2. The life value rule causes the lattice structure to completely decay.
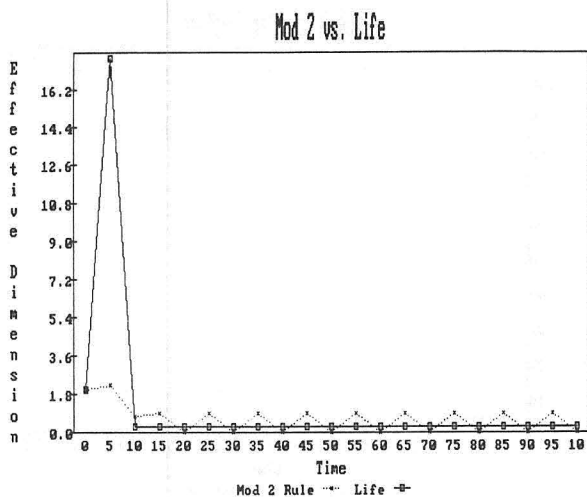
Mod 2 vs. Life

Figure 20: A comparison of the effective dimensionality for the mod 2 rule and life rule. Note that, unlike the mod 2 rule, the life rule leads to final stability.

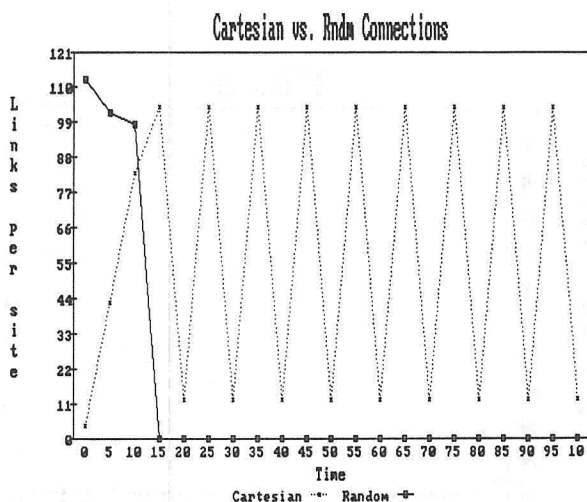## Cartesian vs. Rndm Connections

Figure 21: Here is a depiction of the evolution of the number of links per site for two systems, one which starts off with a four-neighbor Cartesian lattice, the other with a randomly connected network. The link rules and value rules are the same for both, but still there is a qualitative behavioral difference. The initially random lattice decays from over 100 connections per site to 0.
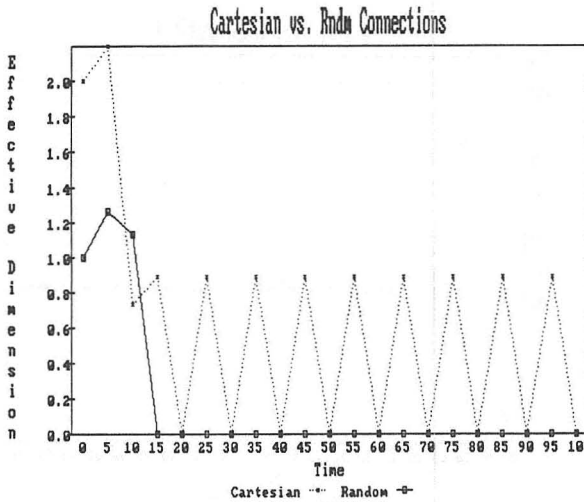
Figure 22: For the random lattice, the effective dimensionality also decays.

Finally, we examined the results of applying purely a coupler or purely a decoupler. In figure 23 we compare the results of the mod 2 and step function value rules on an initially Cartesian lattice for a pure coupler (no decoupler). Here we find that the mod 2 rule leads to rapid expansion until all of the sites are coupled. The value density assumes a periodicity of four (in contrast to two when the decoupler is added). For the step function rule, a stable state is reached with less than half of all sites connected. The value density decays to zero. In figure 24, the mod 2 rule is compared to life. Life also leads to a final stable state, but yet simpler, with only 70 connections per site.

In figure 25, the question of the effect of the choice of initial value seed is considered. A random initial distribution of vales is compared to a set of three ones (a "blinker") for the value rule life. In the former case, there is an expansion to a much larger number of links per site, whereas in the latter case the lattice structure remains stable with eight links per site.

In figures 26 and 27 a pure decoupler is utilized, with the effects of the link rule compared. Note here that the link rule prevents complete decay of the lattice down to zero links per site.

In figure 28, a pure decoupler is utilized and the mod 2 and life rules are compared. Note that both rules allow for decay to a stable state, though in the case of life there is an early period of growth in dimensionality and a final effective dimensionality of 3 and in the mod 2 case there is pure decay of effective dimensionality down to a value of less than 1. Thus, life induces final lattice structures that are far more complex.
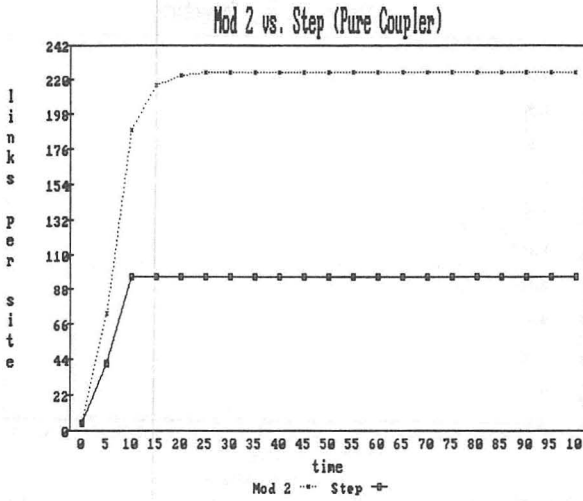
Mod 2 vs. Step (Pure Coupler)

Figure 23: Here we consider the case of a pure link coupler, without a decoupler. We compare the mod 2 and step function rules, and find that, though both lead to a stable state, in the step function case the stable state is much simpler.
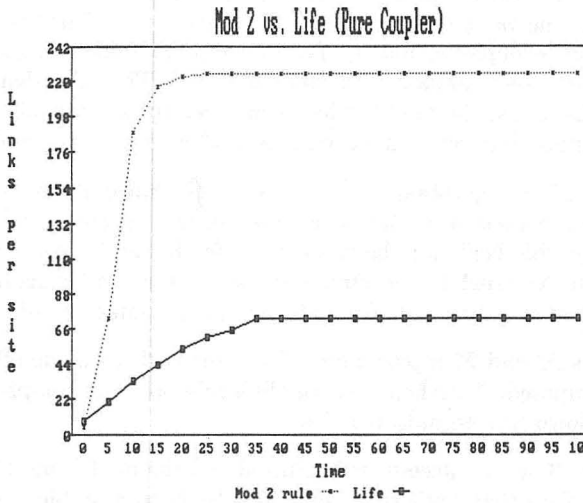
Mod 2 vs. Life (Pure Coupler)

Figure 24: Comparing mod 2 versus life for a pure coupler, we find that the stable state reached by life has far fewer connections.
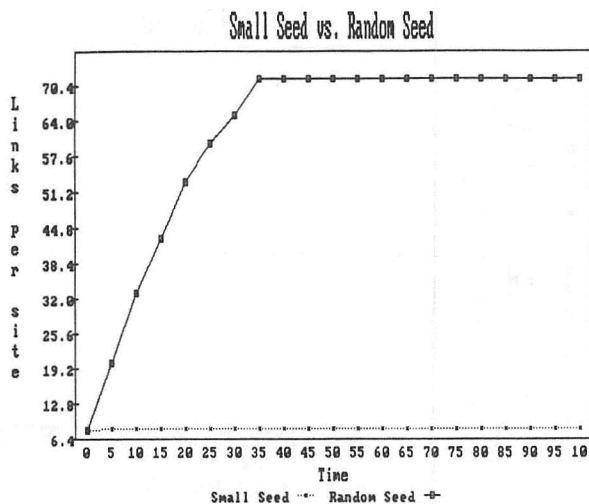
Figure 25: Here we see that even the choice of initial value seed can affect the final outcome. For the value rule life we compare the evolution of a random configuration of ones and zeroes to a "blinker": three ones in a row. There is a sharp difference in the final results for links per site.
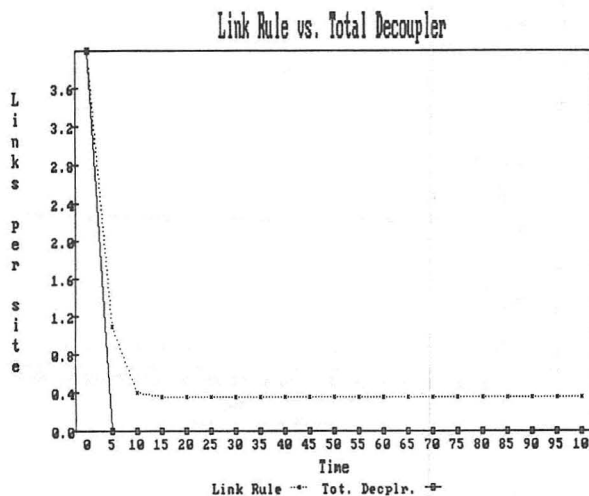

Figure 26: We compare the use of a link rule decoupler to that of complete decoupling of all links. Note that the link rule provides stability in the link structure.
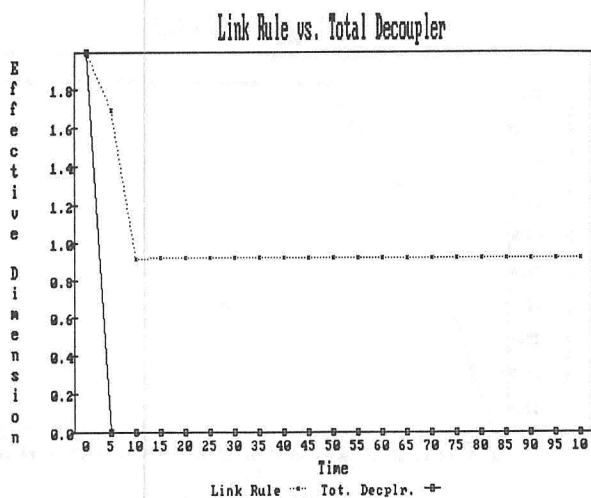
Link Rule vs. Total Decoupler

Figure 27: Similarly, the final effective dimension is different for a link rule versus a complete decoupler.
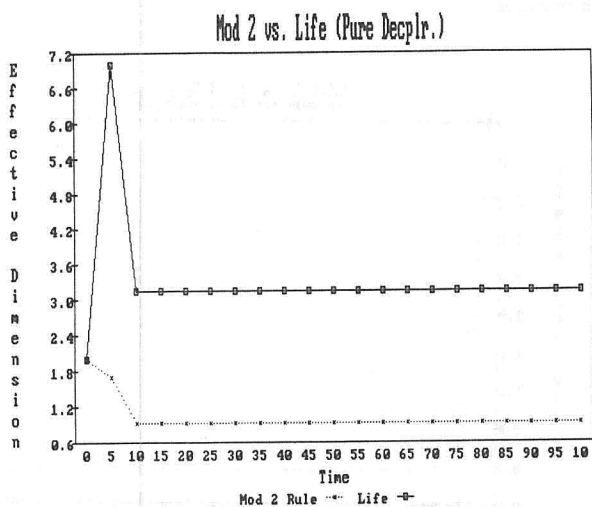
Mod 2 vs. Life (Pure Decplr.)

Figure 28: Comparing the evolution of the effective dimension for both the mod 2 rule and life for a pure decoupler, we see a substantial difference; for mod 2, the dimension exhibits monotonic decay, whereas for life there is a period of dimensional growth.

4. Effects of changing probabilities

It is interesting to examine how the introduction of link rules alters the dynamics of an ordinary cellular automaton. One way to study this transition is to introduce the link couplers and decouplers in a probabilistic manner. Then by increasing the probability of these lattice altering events, one can observe a continuous transformation of the system from one cellular automaton to another.

Physically, the idea is to mimic the effect of random but value dependent lattice alterations. In a lattice gas, for instance, components might, with some probability, lose contact or gain contact. In other words, next-nearest neighbors might have a certain chance of becoming neighbors if they are "aligned," but have no chance if they are not "aligned." In the case of genetic networks, new connections might be made (with a given probability) only if both genes are active, while connections might be severed if both sites are inactive.

With these ideas in mind, we define new probabilistic couplers and decouplers: Probabilistic Boolean decouplers are defined as follows:

$$
\begin{aligned}
&\text{Given} \quad l_{ij}^t = 1, \text{ then } l_{ij}^{t+1} = \Psi_p(l_{ij}^t), \\
&\text{where} \quad \Psi_p = 1 - \delta(v_i^t + v_j^t, 0)[P(D) > r], \\
&\qquad P(D) \text{ is the decoupler probability,} \\
&\qquad \text{and } r \text{ is a random number between 0 and 1.}
\end{aligned}
\tag{4.1}
$$

Therefore, two linked sites i and j are decoupled with probability $P(D)$ if and only if the sum of their site values is zero.

Boolean couplers are defined in the following manner:

$$
\begin{aligned}
&\text{Given} \quad l_{ij}^t = 0 \wedge m_{ij}^t = 1, \\
&\text{then} \quad l_{ij}^{t+1} = \omega(l_{ij}^t), \\
&\text{where} \quad \omega = \delta(v_i^t + v_j^t, 2)[P(C) > r], \\
&\qquad P(C) \text{ is the coupler probability,} \\
&\qquad \text{and } r \text{ is a random number between 0 and 1.}
\end{aligned}
\tag{4.2}
$$

Thus, two unlinked sites i and j have the probability $P(C)$ of becoming linked if and only if they are next-nearest neighbors and the sum of their site values is two.

Let us now see how changing the probability of linking or delinking affects automata dynamics. In figures 29–31, we see an example of how a probabilistic decoupler changes the lattice and value dynamics of a randomly seeded, randomly connected lattice with mod 2 value rule. By altering the probability of decoupling from 0% to 50% to 100%, the final behavior of the value density is altered from that of small fluctuations about 0.5 to a decay to a small static value (around 0.5). Thus changing $P(D)$ causes a qualitative change to take place in the automaton dynamics.

We have investigated a wide range of probabilistic topological automata, altering the value rule, initial lattice structure, initial value seeding and the probabilities $P(D)$ and $P(C)$. We have utilized the mod 2, step function and
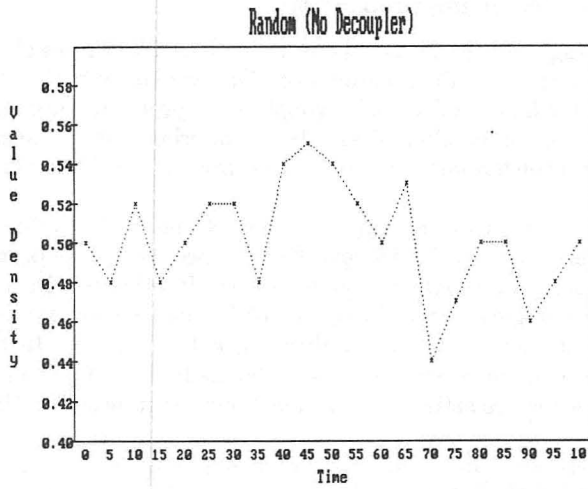
Figure 29: Here is a probabilistic automaton in which the probability of decoupling is set to be zero. Note the random fluctuations in value density.
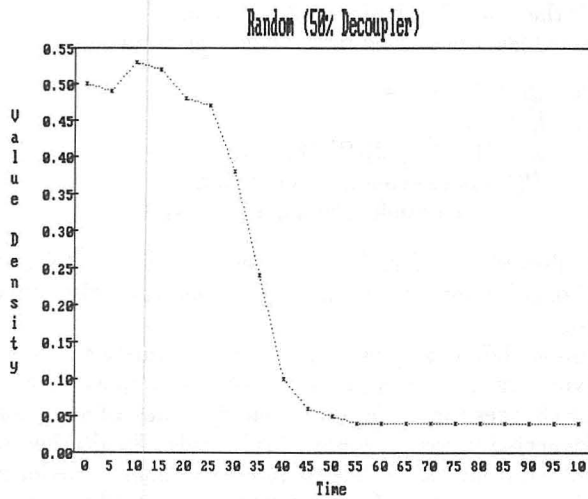

Figure 30: After changing the probability of decoupling to 50%, the value density behavior exhibits monotonic decay.
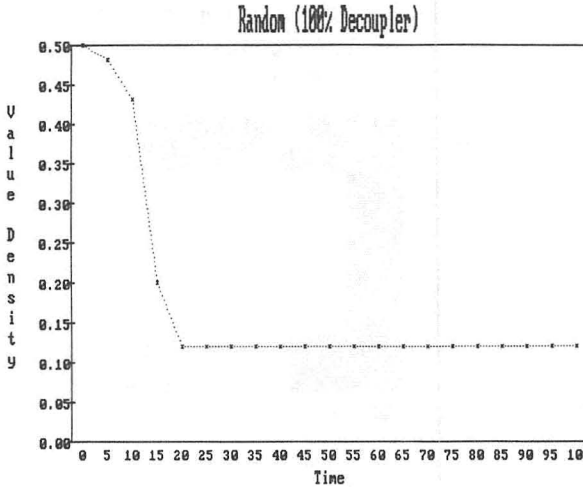
Figure 31: When the probability of decoupling is set to 100%, there is a rapid decay in the value density.

life value rules, Cartesian and random lattices, and random and designated seeding for $0 \leq P(D) \leq 1$ and $0 \leq P(C) \leq 1$. Our results are summarized in figures 32–39 for the value density, links per site, and effective dimension of these automata. We have found a large variety of behavior including lattice growth, decay to a stable state, growth followed by decay, decay to a periodic state, and small fluctuations about a stable lattice.

It is interesting to examine, for instance, the application of probabilistic decouplers and couplers to a Cartesian four-neighbor lattice, seeded randomly and altered with the mod 2 value rule. If one starts off with a standard (no link changes) cellular automaton and gradually increases $P(C)$, one finds that a significant transformation takes place. When $P(C)$ is zero, the value structure oscillates between 30 configurations. However, when $P(C)$ increases from 0.5 to 0.75 to 1, a bifurcation in the value behavior takes place with the final value density altering from stable to having a periodicity of 2 to having a periodicity of 4. The lattice structure itself experiences more and more rapid growth, while the effective dimension approaches zero.

Then, if one increases $P(D)$, the value behavior becomes altered. The final value density first experiences small alterations. Then, as $P(D)$ approaches 1, the final value density decays to 0. Meanwhile, the lattice growth slows down. Eventually, for $0 < P(C) < 0.5$, $0.5 < P(D) < 1$, the lattice structure decays while the effective dimension reaches a state in which it undergoes small fluctuations. Thus, the lattice structure approaches a final decayed state with a small final effective dimension.

One can compare this behavior to that for other value rules. In each of the cases there is a different qualitative sort of behavior, as one can see in

PROBABILISTIC UPDATE
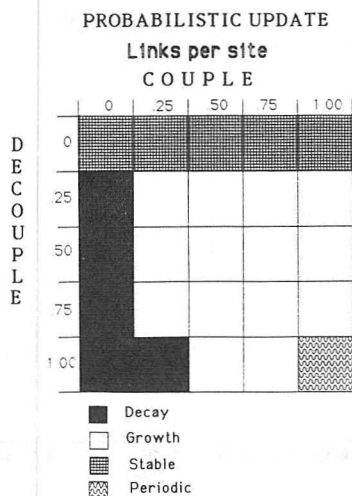**Links per site**
COUPLE

Figure 32: Here are the results for the links per site behavior for a purely probabilistic update. The probability to couple is presented horizontally; the probability to decouple, vertically. Note that for a pure decoupler there is complete decay; for a pure coupler, growth to a stable state, and for a mixed set of couplers, slower growth.
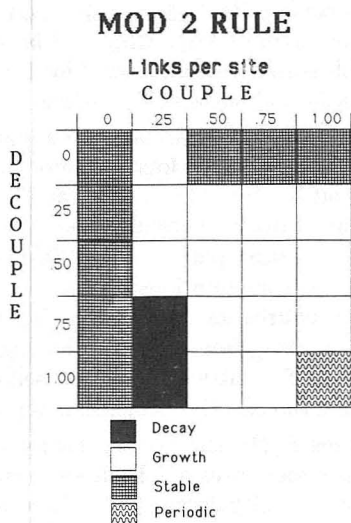
MOD 2 RULE

Links per site
COUPLE

Figure 33: For the mod 2 rule, for a probabilistic topological automaton there are four distinct phases: growth to stability, decay to stability, incomplete growth, and incomplete decay. Note that unlike the pure probabilistic case, these decaying structures reach a stable (not null) final state.

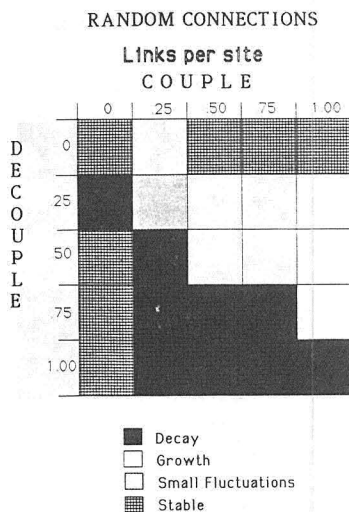RANDOM CONNECTIONS

Links per site

Figure 34: This chart represents the same value and link rules as figure 33. However, here the initial lattice structure is random, not Cartesian. Note that this makes a strong difference in the phase structure of the chart. Also note that for coupler probability 25% and decoupler probability 25%, there exists a new phase- small oscillations about a fixed structure.
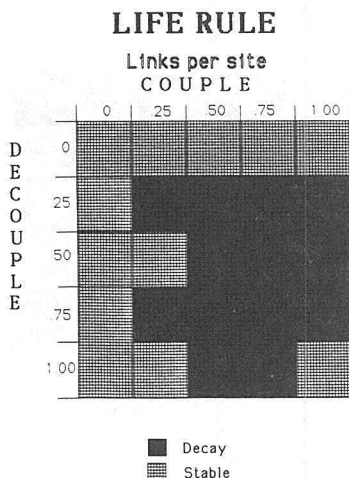
LIFE RULE

Links per site

Figure 35: This chart depicts the results when the coupler and decoupler probabilities are altered for the life value rule. Note that most of the end states are stable, with the rest decaying to stability.

STEP FUNCTION RULE

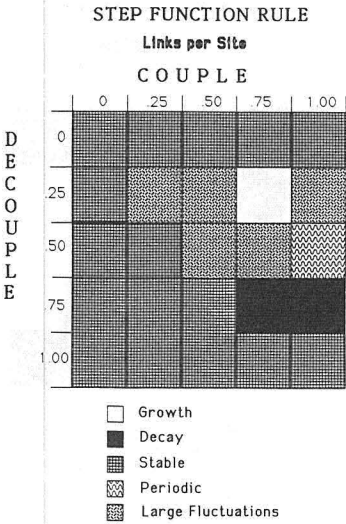**Links per Site**

C O U P L E

Figure 36: Here is the phase structure for the step function value rule. Note the rich variety of behaviors for very small alterations in the decoupler and coupler probabilities.
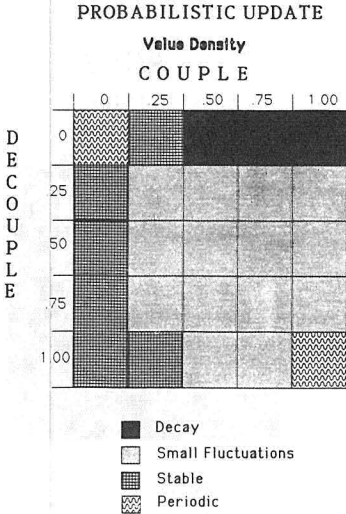
PROBABILISTIC UPDATE

Value Density

C O U P L E

Figure 37: This is the value density behavior for a purely probabilistic updating.
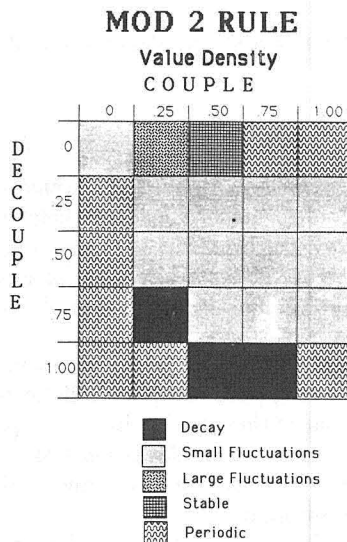
MOD 2 RULE

Value Density
COUPLE

Figure 38: Here is a depiction of the value density behavior for the probabilistic link rules and the mod 2 value rule. Note the differences with the previous diagram. In this diagram, one can examine the bifurcation of the value structure as the coupler probability is increased (with the decoupler probability set to zero.) The value density starts out stable, bifurcates to period 2, then further bifurcates to period 4.

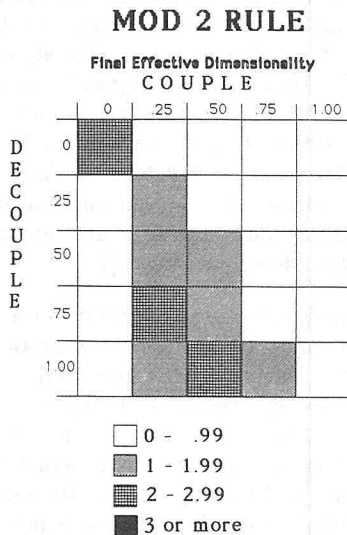## MOD 2 RULE

Final Effective Dimensionality
COUPLE

Figure 39: Here is the range of final effective dimensionalities for the mod 2 rule. Note that by altering the link rule probabilities one can cause a sort of "dimensional reduction," decreasing the final dimensionality by as much as 2.

figures 32–39. Thus there is a strong dependence of the dynamics on the value rule.

5. Conclusion

In this preliminary survey, we have found a wide range of topological automaton behavior for our Boolean link rules. This behavior includes long-term growth, decay, growth followed by decay, and decay to a simpler state of lower dimensionality. We have found that this behavior is strongly influenced by the choice of initial value configuration, initial link structure, and the value rules.

We have also explored a modified version of this model, in which the link couplers and decouplers are applied in a probabilistic manner. It is of interest to note that by altering one of these probabilities as a parameter one can force a cellular automaton to experience bifurcation from stability to periodicity (and perhaps even to chaos). One can also induce a dimensionality altering transition for a cellular automaton.

This result lends itself to speculation on physical and biological applications. The fact that link rules can alter automaton dimensions suggest use of these models to depict compactification in a cellular automaton based quantum field theory, such as the one developed by Svozil [10].

In theoretical biology, there are some parallels between topological automata and genetic network theory, such as that developed by Kauffman [7]. Both involve nodes, connections, and a changing link dynamics.

In Kauffman's scheme, an idealized chromosome is considered in which genes, depicted as points, regulate each other through connections presented in a "wiring diagram." This "wiring diagram" model serves as a cellular automaton in which site values are either 0 or 1 (inactive or active) and the value rules represent a simple Boolean dynamics. However, unlike conventional cellular automata, the link structure of these networks is dynamic. Genetic mutations, represented as random fluctuations, can alter the network structure. Kauffman considers, as a first approximation, that these link changes are purely random, but states [11]:

> "Fully random directed graphs constitute a beginning point for studies of the connectivity features of mutating genetic systems, but are only a background. The actual ways chromosomal mutations 'scramble' the genetic regulatory system are not fully random in an equivalent sense. For example, a major mechanism creating novel regulatory connections presumably involves duplication of a sequence and its dispersion... to new positions in the genome. The probabilities of generating a new duplication are almost certainly not independent of the number and sequence of that sequence already present."

Perhaps Boolean topological automaton theory, particularly the probabilistic automata considered earlier, could play a part in realizing this goal.

Thus mutations (read: decouplers and couplers) could be both probabilistic and value dependent (as in the link rules considered earlier). Probabilistic link rules could then model the evolution of genetic regulatory systems.

Other applications for TAs might also be considered. Kauffman has suggested that it may be of interest to explore applications for these models in economic theory [12]. Hopefully, further study of topological automata will yield more information on these connections. A computer study of the application of TAs to genetic nets is currently in progress.

Acknowledgments

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