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Upper Bound on the Number of Cycles in Border-Decisive Cellular Automata

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Abstract. The number of stable states of any one-dimensional k-state border-decisive cellular automaton on a finite lattice with periodic boundary conditions is proved to be bounded by k^R and the number of cycles of length p is bounded by $\frac{1}{p}\sum_{q|p}\mu(p/q)k^{qR}$, where $(R+1)$ is the number of neighbors and μ is the Möbius function.

1. Introduction

We consider a deterministic cellular automaton (CA) [1] on a one-dimensional array of N sites. At each site i, there is a dynamical variable whose value at time t will be denoted by $a_i^{(t)}$. Here $a_i^{(t)}$ belongs to a finite set S with k elements. The state of the CA with N sites can be described by an N-component vector $\vec{A}^{(t)}$ ($\vec{A}^{(t)} \in S^N$), whose the ith-component is $a_i^{(t)}$. The time evolution is given by $\vec{A}^{(t+1)} = \Phi(\vec{A}^{(t)})$ where Φ is defined by a local deterministic rule ϕ: $S^{r+1} \to S$ of the form

$$a_i^{(t+1)} = \phi[a_{i-r_1}^{(t)}, a_{i-r_1+1}^{(t)}, \dots, a_{i+r_2}^{(t)}], \tag{1.1}$$

where r_1 and r_2 are the number of neighbors to the left and to the right respectively, and r_1+r_2+1 ($\equiv R+1$) is the total number of neighbors. The time evolution of a CA on a finite lattice can be represented by a graph, called the state transition diagram, whose nodes correspond to the states of the CA; directed arcs connecting the nodes represent transitions between the states. In general, the complete topology of the state transition diagram is difficult to determine except for a special class of CA, viz., additive CA [2,3]. In this note we give upper bounds for the number of stable states and of cycles of a more general class of CA, border-decisive CA.

Definition 1. *A one-dimensional N-site k-state deterministic CA is said to be border-decisive if its local rule ϕ satisfies*

$$\phi(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2}) \neq \phi(a_{i-r_1}, a_{i-r_1+1}, \dots, a'_{i+r_2}) \text{ if } a_{i+r_2} \neq a'_{i+r_2} \tag{1.2}$$

with $r_2 > 0$ and for any fixed values of $(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1})$, or

$$\phi(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2}) \neq \phi(a'_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2}) \text{ if } a_{i-r_1} \neq a'_{i-r_1} \tag{1.3}$$

with $r_1 > 0$ and for any fixed values of $(a_{i-r_1+1}, a_{i-r_1+2}, \dots, a_{i+r_2})$.

Note that we impose periodic boundary conditions and hence, the site indices are computed modulo N. In words, for a border-decisive CA rule, if all the neighbors of site i except the one on the "border" (leftmost or rightmost) are fixed at time t, then $a_i^{(t+1)}$ will be different for different values of the border site at t. The class of border-decisive CA has been investigated [4] also under the name of toggled rule [5] and left and right permutive rule [6].

2. Results

Theorem 1. *The number of stable states of any border-decisive CA has an upper bound $n_1 = k^R$.*

Proof. Let ϕ be a CA rule satisfying (1.2). (The same proof goes through for rules satisfying (1.3).) For any fixed $(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1})$, ϕ_1: $S \to S$ defined by

$$\phi_1(a) = \phi(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1}, a) \tag{2.1}$$

is a one-to-one function. Since S is finite, ϕ_1 is also onto and, therefore, ϕ_1 is invertible. Now suppose $\vec{A}$ is a stable state, i.e., $\Phi(\vec{A}) = \vec{A}$. The values of a string of R sites $(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1})$ for arbitrary i determine the entire set of stable states as follows: For arbitrary but fixed value of this string we can compute a_{i+r_2} from

$$a_{i+r_2} = \phi_1^{(-1)}(a_i). \tag{2.2}$$

Knowing a_{i+r_2}, a_{i+r_2+1} can be determined, and so on. Since we have imposed periodic boundary conditions we eventually return to the original string of R sites. A stable state ensues if and only if the values thus obtained are identical to those assigned initially to the string. Since there are k^R choices for the values of the string the number of stable states cannot exceed k^R. ∎

Theorem 2. *The number of cycles of length p of any border-decisive CA has an upper bound $n_p = (k^{pR} - \sum_{q|p, q\neq p} q n_q)/p = \frac{1}{p}\sum_{q|p} \mu(p/q) k^{qR}$.*

Proof. Consider a state $\vec{A}$ on a cycle of length p, i.e., $\Phi^p(\vec{A}) = \vec{A}$. Since ϕ_1 (which is assumed to satisfy (1.2)) is invertible, the values of a rectangular $R \times p$ patch of the space-time pattern $[a_{i-r_1}^{(t)}, ..., a_{i+r_2-1}^{(t)}]$ for $t = 1, 2, ..., p$ and arbitrary i determine the entire set of states on cycles of length p. First $a_{i+r_2}^{(t)}$ for $t = 1, 2, ..., p$ can be determined uniquely by successively using

$$a_{i+r_2}^{(t)} = \phi_1^{(-1)}[a_i^{(t+1)}]. \tag{2.3}$$

(Note that $a_j^{(t+p)} = a_j^{(t)}$ for all j and t due to the periodicity in time.) Then the values of the rest of the sites can be ascertained by iterating the above procedure. As before, the periodic boundary conditions lead to consistency requirements. Since the initial patch can assume k^{pR} different configurations, one easily obtains the weaker bound k^{pR}/p. (If b is a bound for the number of states on cycles of length p, then b/p is the corresponding bound for the number of cycles of length p.) However, some of the k^{pR} configurations for the initial patch have period q ($< p$) where $q \mid p$ (q is a divisor of p). Clearly, for such configurations of the initial patch if the consistency conditions are satisfied one obtains a cycle of length q because the periodicity of the initial patch guarantees $a_{i+r_2}^{(t)} = a_{i+r_2}^{(t+q)}$ for $t = 1, 2, ..., q$. If the consistency conditions are not satisfied one does not obtain a cycle of length either p or q. So we can, in fact, subtract the number of such configurations (which can be expressed in terms of n_q) from k^{pR} and thus obtain the maximum number of states which can be on cycles of length p; dividing by p yields the upper bound for the number of cycles. If we write the bound iteratively it is easy to see that the first expression for the upper bound in Claim 2 is obtained. Alternatively, the bound can also be expressed in terms of the Möbius function μ, where $\mu(1) = 1$; $\mu(n) = 0$ if n has a squared factor; and $\mu(p_1 p_2 ... p_l) = (-1)^l$ if all the primes p_1, p_2, ..., p_l are different [7]. ∎

This upper bound on the number of cycles of a given length depends only on k and R. It is independent of the size N of the CA as long as N is finite. For $k = 2$, $R = 2$, the bounds on number of cycles of lengths up to ten are 4, 6, 20, 60, 204, 670, 2340, 8160, 29120, and 104754 respectively. In addition, it is easy to see that the following statements are true: Let N be the smallest size of a given border-decisive CA such that the upper bound n_p for the number of cycles of length p is reached (this is possible only if $pn_p \leq k^N$); then the same CA will reach the same bound for size $M > N$ if and only if M is a multiple of N. (Note that there is no restriction on M being finite.) On the other hand, if the actual number of cycles of length p is $m < n_p$ for a CA with N sites, then the number of cycles of length p for the same CA of size M is no less than m when M is a multiple of N. Thus by calculating the number of cycles for a CA for relatively small sizes one can get the lower bound for, and sometimes even the exact value of, the number of cycles for very large sizes. (This is possible only for sizes which have relatively small divisors.) For example, for Rule 90 when $N = 3$ there are four 1-cycles. We can immediately deduce that all sizes N which are multiples of 3 have four 1-cycles. For $N \leq 30$ the maximum number of cycles of length p for $p \leq 8$ are: 4, 6, 20, 60, 51, 670, 36, and 8160 [2]. These first occur for N = 3, 6, 15, 12, 17, 30, 9, and 24 respectively and hence, the upper bound is reached for p = 1, 2, 3, 4, 6, 8. However, for Rule 30 and $N \leq 17$ the corresponding maximum number of cycles are: 3, 0, 4, 7, 4, 0, 15, 1 [8].

We will now determine, for a given k and R, how many of the $k^{k^{R+1}}$ possible CA rules are border-decisive. Let B_1 be the set of rules satisfying (1.2), B_2 that satisfying (1.3), and B be the set of all border-decisive rules. We have $B = B_1 \cup B_2$ and $|B|$, the cardinality of B, satisfies

$$|B_1| \leq |B| = 2|B_1| - |B_1 \cap B_2| \leq 2|B_1| \tag{2.4}$$

where we have used the symmetry relation $|B_1| = |B_2|$. It is easy to compute $|B_1|$. Notice that for fixed $(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1})$, ϕ_1: $S \to S$ defined by (2.1) is actually a permutation of S, i.e., ϕ_1 is an element of the permutation group of k elements S_k. Since a border-decisive rule is specified by assigning a ϕ_1 to each value of $(a_{i-r_1}, a_{i-r_1+1}, \dots, a_{i+r_2-1}) \in S^R$, it is a mapping from S^R to S_k. On the other hand, it is obvious that each mapping from S^R to S_k gives a border-decisive CA rule. From $|S_k| = k!$ and $|S^R| = k^R$, we have

$$|B_1| = (k!)^{k^R}. \tag{2.5}$$

Comparing with the total number of possible CA rules, $\mathcal{R} = k^{k^{R+1}}$, we have,

$$\left(\frac{k!}{k^k}\right)^{k^R} \leq \frac{|B|}{\mathcal{R}} \leq 2\left(\frac{k!}{k^k}\right)^{k^R}. \tag{2.6}$$

In the cases one usually considers, where both k and R are small, the fraction of border-decisive rules is not negligible. For small k, $|B_1 \cap B_2|$ can be easily calculated. $|B_1 \cap B_2| = 2^{2^{R-1}}$ for $k = 2$ and $|B_1 \cap B_2| = 12^{3^{R-1}}$ for $k = 3$. Given $|B_1 \cap B_2|$ and $|B_1|$, the number of border-decisive rules $|B|$ can be computed from (2.4). For $k = 2$, $R = 1$, $|B| = 6$ ($\frac{|B|}{\mathcal{R}} = 0.375$), for $k = 2$, $R = 2$, $|B| = 28$ ($\frac{|B|}{\mathcal{R}} \approx 0.109$), for $k = 3$, $R = 1$, $|B| = 420$ ($\frac{|B|}{\mathcal{R}} \approx 0.021$).

3. Some generalizations

The results of the last section can be extended to the in-degree of any node (i.e., the number of predecessors) of the state transition diagram and to higher-order rules. We state the following corollaries without further explanation since the proofs are similar to the ones given above.

Corollary 1. *The in-degree of any node of the state transition diagram of a border-decisive CA with k states and $R+1$ neighbors is bounded above by k^R.*

Definition 2. *A deterministic CA rule Φ of order s is given by*

$$\vec{A}^{(t+1)} = \Phi[\vec{A}^{(t)}, \vec{A}^{(t-1)}, ..., \vec{A}^{(t-s+1)}], \tag{3.1}$$

where Φ is defined by the local rule ϕ: $S^{\bar{R}+1} \to S$ of the form

$$a_i^{(t+1)} = \phi[a_{i-r_{1_1}}^{(t)}, ..., a_{i+r_{2_1}}^{(t)}, ..., a_{i-r_{1_s}}^{(t-s+1)}, ..., a_{i+r_{2_s}}^{(t-s+1)}]. \tag{3.2}$$

Here $\bar{R} + 1 = \sum_{i=1}^{s}(r_{1_i} + r_{2_i} + 1)$. A deterministic CA rule of order s on a finite lattice is border-decisive if the function ϕ in (3.2) is a one-to-one function of the argument $a_{i+r_{2_j}}^{(t-j+1)}$, where $r_{2_j}(>0)> r_{2_i}$ for all $i \neq j$, for each set of fixed values of the other $\bar{R}$ arguments, or if ϕ is a one-to-one function of the argument $a_{i-r_{1_j}}^{(t-j+1)}$, where $r_{1_j}(>0)> r_{1_i}$ for all $i \neq j$, for each set of fixed values of the other $\bar{R}$ arguments.

Let $R = r_{1_j} + r_{2_k}$ where $r_{1_j} = \max_i(r_{1_i})$ and $r_{2_k} = \max_i(r_{2_i})$. We have, then:

Corollary 2. *The number of stable states of a finite k-state s-order border-decisive CA is bounded above by $n_1 = k^R$ and the number of cycles of length p is bounded above by $n_p = (k^{pR} - \sum_{q|p,q\neq p} qn_q)/p = \frac{1}{p}\sum_{q|p}\mu(p/q)k^{qR}$.*

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